THE
DOCTRINE
OF
FLUXIONS:

Not only explaining

The ELEMENTS thereof,

But alſo its

APPLICATION and USE

In the ſeveral Parts of MATHEMATICS and Natural PHILOSOPHY.

Ornari res ipſa negat contenta doceri.

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The PREFACE.

TO say any thing in Praiſe of the Method of Fluxions, or of its Dignity and Rank among the Mathematical Sciences, would be as needleſs as to deſcribe the Excellency of bright Sun-ſhine above the twinkling Light of the Stars; ſince any one who is acquainted with the Sciences will allow it to be a Method of Calculation incomparably ſuperior to all other Methods that ever were known or found out; and beyond which nothing further is to be hoped or expected. It lends it's Aid and Aſſiſtance to all the other Mathematical Sciences, and that in their greateſt Wants and Diſtreſſes: It opens and diſcovers to us the Secrets and Receſſes of Nature, which have always before been locked up in Obſcurity and Darkneſs. To this all the noble and valuable Diſcoveries of the laſt and preſent Age are entirely owing: And by this Method Sir Iſaac Newton, *the worthy Inventor, determined and ſettled the Syſtem of the whole viſible World.*

The Uſe and Application of FLUXIONS *are exceedingly extenſive; for Example, in Trigonometry, it teaches the Computation of Sines, Tangents and Secants; in Arithmetic, the Calculation of Logarithms; in Geometry, drawing Tangents to Curves, finding their Curvatures, their Lengths, and Quadratures, the Surfaces and Solidities of Bodies; in Mechanics and Philoſophy, the Inveſtigation of the Centers of Gravity and Oſcillation, the Vibration of Pendulums, the Laws of Centripetal Forces, the Times, Velocities, and Spaces deſcribed by Bodies acted upon by any Forces, the Motions and Reſiſtances of Bodies in Mediums,* &c. *Theſe are ſome of the numberleſs Inſtances, wherein Fluxions are applied with ſuch wonderful Succeſs. And though ſome few of theſe may be (and actually have been) hammer'd out with great Labour and Difficulty by other Methods; yet the Proceſs of none of them can in the leaſt be compared with that Beauty, Simplicity, and charming Elegance, with which*

the Method of Fluxions performs all these Things. In short, the Method of Fluxions is capable of resolving such Difficulties as raise the Wonder and Surprise of all Mankind, and which would in vain be attempted by any other Method whatsoever. So that it is justly esteemed the greatest Work of Genius, and the noblest Thought that ever entered the human Mind.

The Method of Fluxions is founded upon this most simple and obvious Principle, viz. *that any Quantity may be supposed to be generated by continual Increase, after the same Manner that Space is described by local Motion. The great and noble Inventor tells us, that in this Method* he considers Things as generated by continual Increase, after the Manner of a Space which a Thing or Point in Motion describes. *Now the Conception of this is exceeding easy and natural; for we every Day see with our own Eyes all kinds of Lines and Figures described by the Motion of Bodies: This Principle then will be easily admitted. And further, since we also see by Experience, that these very Lines and Figures are described, some with greater Degrees of Velocity, some with less, some with Motions continually accelerated or retarded, and some with uniform Motions: We shall easily understand that any one of these Lines or Spaces has in every Point of it's Description a certain Degree of Increase determinate in it self, and peculiar to that Point, and which is the same with the Velocity of the Thing that describes it. And to determine this Velocity, or this Degree of Increase, in any given Point of the generated Quantity, is the same Thing as finding the Fluxion of a proposed variable Quantity, and is the Foundation of all the Arithmetick of Fluxions. And to determine this truly is of the greatest Consequence for establishing the Theory.*

Many Disputes and Objections have been advanced against the Truth of the Method of Fluxions; and amongst these Disputants, as it commonly happens, those have been the most inveterate, who understood the least of the Matter. To answer all the Cavils that have been offered will be of little Consequence in any thing, and of none at all for

settling

ſettling the true Notion of Fluxions. Therefore, inſtead of that, I ſhall, by following Nature as cloſely as I can, endeavour to give the unprejudiced Reader a clear and true Idea thereof; and then perhaps he will be able to judge for himſelf, whether the Principles of this noble Art be capable of Demonſtration or not. In order to this, let us aſſume what has been before laid down, that any generated, flowing Quantity is analogous to a Line deſcribed by a moving Point, and that the Velocity of this Point in any place repreſents the Fluxion of that Quantity in the correſpondent Place of the Fluent; now I ſhall conſider the Generation of this Line, inſtead of the Fluent, as being more eaſily underſtood.

It is the general Practice in Mechanics, to meaſure the Velocity of a Body by the Space uniformly deſcribed in a given Time. For Velocity being that *by which a Body is carried through a given Space in a given Time, therefore Velocity muſt be looked upon as the proper efficient Cauſe of the Space deſcribed; and the Space deſcribed the adequate Effect of that Cauſe. Now ſuppoſe a right Line deſcribed with any ſort of Velocity, accelerated, or retarded, at Pleaſure, and that we would enquire what is the Velocity of it in any given Place. If we take a ſmall Part of the Line, which the moving Point deſcribes juſt before it arrives at that Place, and call it an Increment, and ſuppoſe it to be deſcribed in a very ſmall given Time; then this Increment will nearly meaſure the Velocity of the deſcribing Point at the place propoſed, and is ſufficient to give a vulgar Notion of the Degree of Velocity required. Now if this right Line was deſcribed uniformly, this would accurately meaſure the Velocity. But ſince that Increment is deſcribed with a Velocity, by ſuppoſition, continually variable, therefore this Notion we have here obtained is to be corrected; the firſt Notions we get of any Subject are generally incorrect, and demand a nicer View, and a more accurate and philoſophical Examination, before we can acquire Notions that are perfect and adequate. Here then it will be very evident, if we take ſtill a leſſer and a leſſer Increment, by which the Velocity*

is measured, as the Point still draws nearer the proposed Place; we approach nearer and nearer to a uniform Velocity, till the Difference be less than any assignable: And this Increment will differ from the true Measure of the Velocity, by less than any given Difference: And as this Increment continually diminishes, till at last it vanish, it approaches continually to that Measure, till the Difference vanishes with it.

Now although by diminishing the Increment at Pleasure we can approach within any Degree of Exactness to the Velocity required, yet since no Increment can be taken so small, but it is still further divisible ad infinitum; *and since the Velocity is by Supposition continually variable, it is plain, there can be no two Points of this Increment in both which the Velocity is accurately the same. It is therefore most manifest, that the Velocity here enquired after is peculiar to one only indivisible Point; and that Point is the Place where the Increment ends, or vanishes into nothing. Here then we see plainly, that the Velocity in any given Point of the Line described (or, which is the same thing, that the Fluxion in any given Point of a generated Quantity) has a certain, fixed, determinate Value, proper to that Point of it alone: And this furnishes the Mind with that accurate abstract Idea, which we ought to form of this Velocity or Fluxion. And here we may observe, that this Degree of Velocity (or Fluxion) we have been here considering, and which continues but a Moment, differs from the same Degree of Velocity (or Fluxion) which continues for any given Time, and by which a given Space is actually described; these, I say, differ no otherwise than as a Cause in* POWER *differs from a Cause in* ACT.

Here a metaphysical disputant may demand, how it comes to pass, that any Velocity which continues *for no Time at all, can possibly describe any Space at all; or whether its Effect be absolutely nothing, or an infinitely small Quantity, or what it is. Here then it is, that our Reason is at a Stand, and the human Faculties are quite confounded, lost, and bewildered. We are puzzled*

and perplexed by endeavouring to examine into the Nature of we do not know what, *nor whether it is ſomething or nothing: And at beſt is ſome ſuch ſubtile, fleeting Thing, as the Mind can lay no Hold on, nor form any Idea of. Now whether ſuch ſubtile Queſtions will be ever determined, or not, yet there is one Refuge for us,* viz. *that it is nothing at all to our Purpoſe what they are: And therefore we may ſafely leave theſe deep Speculations to thoſe that have more Buſineſs with them. The Method of Fluxions has no Dependence on theſe myſterious Diſquiſitions. What I apprehend the Method of Fluxions to be concerned in, is, not what any ſingle abſtract Velocity can deſcribe or generate of itſelf, but what a continual and ſucceſſively variable Velocity can produce in the whole. And here I think no Reaſon can be aſſigned, why a variable Cauſe ſhould not produce a variable Effect, as well as a permanent Cauſe a permanent and conſtant Effect. For ſince every Effect has a co-inſtantaneous Exiſtence with its efficient Cauſe, and is always perfectly connected with it; all the Difference can only be this, that the continual Variation of the Effect muſt always depend on, and be proportional to, the continual Variation of the Cauſe that produces it. And this will always be true, though we have no Ideas at all of the perpetually ariſing Increments, or their magnitude in their naſcent or evaneſcent State, that have ſo much, and to ſo little Purpoſe, confounded and puzzled the mathematical World. And whether we can or we cannot conceive the formal Nature or Manner of exiſting of a thing juſt ariſing out of nothing, or beginning to be; or whether a naſcent or evaneſcent Quantity be any thing or nothing; yet the truth of the Method of Fluxions will ſtand juſt as it did. But theſe ſort of Diſputes have been artfully introduced for no other Purpoſe but to involve the Subject in Obſcurity, to darken the Reader's Judgment, and thereby to miſlead and divert him from purſuing the principal Buſineſs in Hand, that is, from conſidering the proper Evidence on which alone this Doctrine is founded; by inſinuating that the Knowledge*

of

of these things is essential to the very Nature and Foundation of Fluxions: When, it is evident, all that the Method of Fluxions does or ever did propose, being either to determine the Velocity (or Fluxion) wherewith a generated Quantity increases in any given point; or else to sum up all that has been described or generated by such continually variable Fluxion, during any given Time, or to any given Point of the Fluent or generated Quantity. These two things alone are the two Bases on which this noble Structure (the Method of Fluxions) is to be erected. All metaphysical Speculations, of what Nature soever, having no Business here.

I shall now bring an Instance or two out of the Phænomena of Nature, which will help the Reader's Notions a little; and will shew, that what has been said before, concerning the Nature and Idea of Fluxions, is really true, and agreeable to the Nature and Constitution of Things. Let a heavy Body descend through a perpendicular height of $16\frac{1}{12}$ *Feet in one Second of Time, according to the* Gallilean *Hypothesis of Gravity; then at the end of this Second of Time, the Body has acquired a Velocity of* $32\frac{1}{6}$ *Feet in a Second; which therefore is accurately known. Now take any Point A in the right Line, at any given Distance from the Place the Body fell from, and the Velocity which the falling Body has in the Point A may be most accurately computed. But take any Point above A, though at ever so small a Distance, if it be distant at all from A, and the Velocity in that Point will always be something less than in the Point A. And in like manner the Velocity at any Point below A, though indefinitely near it, will be something greater than in A: and therefore it is plain, that to the Point A, there belongs a certain determined Degree of Velocity, which belongs to no other Point in the whole Line, and this is accurately the Fluxion of that right Line in the point A; and is the Velocity with which the Body would proceed if the Force of Gravity should be supposed immediately to cease when the Body arrives at A, and to act no longer.*

Let

Let there be a glaſs Tube, open at both Ends, and whoſe Concavity is of different Diameters in different Places, let it be immerſed in a running Stream of clear Water, ſo that the Water may flow freely through it, and always fill the Tube. Then it is evident, that in different places of the Tube, the Velocity of the Water will be reciprocally as the Squares of the Diameters of the Tube, in theſe places, and will therefore be different. Therefore if you mark any Place in the Side of the Tube, and ſuppoſe a Plane to paſs through the Tube perpendicular to the Axis, or to the Motion of the Water, then the Water will always paſs through this Section with a certain determinate Velocity. But ſuppoſe another Section to be drawn, though ever ſo near the former, then (by reaſon of the ſuppoſed different Diameters) the Water flows through this with a Velocity different from that it did at the former: And therefore that given determinate Velocity belongs only to one ſingle, indiviſible Point, or Section of the Tube, and this is the Fluxion of the Space which the Fluid deſcribes at that Section; and is that uniform Velocity with which the Fluid would *continue to move, if the Diameter continued the ſame through the ſucceeding part of the Tube. Something like this may be obſerved in a River, for there the Velocity is greateſt, where the Dimenſions are leaſt, and leſs where theſe are greater.*

Again, let a hollow Cylinder be filled with Water, and let it flow freely out through a Hole at the Bottom of it. It is well known, that the Velocity of the effluent Water depends on the Height of the Water within the Cylinder; and therefore, ſince the Surface of the incumbent Water continually deſcends without any the leaſt Stop, the Velocity of the effluent Stream will continually decreaſe, till it all be run out. Therefore it is plain, there can be no two Moments of Time, ſucceeding each other ever ſo nearly, wherein the Velocity of the running Water is preciſely the ſame. And therefore the Velocity that the effluent Water has at any given Point of Time, belongs only to that one particular, indiviſible Moment of Time,

and no other: And this is accurately the Fluxion of the Fluid flowing out at that Moment of Time. Now if precisely at that Moment you begin and continue to pour more Water into the Cylinder, so that the Surface of the Water may descend no lower, but keep it's Place; then the effluent Water will also retain it's Velocity, and continue to be the Fluxion of the Fluid as before. Now these are the genuine Effects and Operations of Nature it self; and do, in a manner visibly, confirm the Truth of what has been said of the Nature of FLUXION.

From these Examples and many more that might be produced, it is clear to me, that it is an essential Property of the Fluxion of a generated Quantity, that it does not retain any one determined Value for the least Space of Time whatever; but at the Moment it arrives at that Value, the same Moment it leaves it again; so that it only passes gradually and successively through all the indefinite Degrees contained between the two extreme Values which are the Limits thereof, during the Generation of the Fluent: That is, in case the Fluxion be variable at all; but if it is invariable, the extreme Values, and all the intermediate Degrees, are but one and the same Value. And therefore, although any determinate Degree of Fluxion do not continue at all, yet every Fluent has (intrinsically) in itself, some determinate Degree of Fluxion, at every determinate, indivisible Moment of Time.

It being now, I suppose, made evident, that every generated Quantity has every where a certain Rate *of increasing (called its* Fluxion) *whose abstract Value is determinate in itself, at any determinate Point of that Quantity: Therefore to find out its Value, or its Ratio to any other Fluxion, is a Problem strictly geometrical. It remains to enquire in general how we must compute this Value. And here the only, or at least the most natural Way is, to get the Proportion of the Increments generated by the Fluxions in all Suppositions of Magnitude of these Increments, and from thence collect the Ratio they first begin with. When the Fluxions and*

Moments

Moments of the ſimple literal Quantities are deſigned by proper Symbols; it will be eaſy by the binomial Theorem to find the contemporary Increment of any compound Quantity; for that will be expreſſed in the Form of a Series. Now although it is evident, that this Increment of the compound Quantity is not accurately the Fluxion of it in any determinate Point, becauſe that very Increment is generated by a Fluxion continually variable; yet it is as evident *that it continually approaches to it, by continually diminiſhing the Increments of the ſimple Quantities. Here then will be had in general, the Ratio of the Fluxion of a ſimple Quantity to the Fluxion of that compound Quantity, and in the loweſt Terms, and that as near the truth as we pleaſe, whilſt we ſuppoſe ſome, though very ſmall, Increment actually deſcribed. But ſince the Ratio of theſe Fluxions is required for, and belongs only to, ſome one indiviſible Point of the Fluent, that is, in the very beginning of the Increment, or when there is no Increment at all generated; therefore in this particular Caſe making the Values of the ſimple Increments nothing, which before was expreſſed in general, and then all the Terms wherein they are found will vaniſh, and what is left will accurately ſhow the Relation of the Fluxions, for that ſingle indiviſible Point where the Increment is ſuppoſed firſt to commence, or was required. For this abſtract Value of the Fluxion belongs to no more Points than one of the Fluent; and therefore of Conſequence the Moments muſt be made to vaniſh, after we have ſeen by the continual Diminution thereof, whither the Ratio tends, and what it continually converges to; which will be as viſible to every Body as the very Characters it is written in. And if any one ſhould doubt of the Truth of this, I ſhould for ever deſpair of convincing him of any thing at all. The Increments here muſt neceſſarily be made uſe of, not to determine* their *Magnitude as ſome have abſurdly imagined, but as a* Medium *in our Reaſoning, to diſcover the Quantity of the Cauſe that produces them, they being the continual Effects of the Fluxions; and how can we*

judge of the Force or Efficacy of a Cause, without considering the Effect that it does or could produce: For that like Causes are proportional to their Effects can never be denied, except by those that can deny any thing. This Way of Reasoning and Method of Demonstration then must be exceedingly clear and convincing to all that are duly qualified to examine and consider it, and do not with a most unaccountable Obstinacy, and invincible Prejudice, resolve to yield to no Reason at all, though laid before them as clear as the Sun.

And here it may be worth observing, that in the Process of this Demonstration, the Terms which vanished out of the Increment of the compound Quantity, did plainly arise from, and was generated by the Variation *of the Fluxion of that compound Quantity; and the remaining Quantities alone are those generated by the Fluxion itself.*

The Easiness and Simplicity of this Method of Demonstration is no small Argument for it's Truth and Perfection. The Simplicity of Truth is it's great Beauty. And by this Mark it here proves itself to be the genuine Off-spring of Nature and Truth. But if any Persons will not assent to the Truth of these Principles, I would have them suspend their Judgments, lest they make it appear that they have no Judgment at all. In the mean Time let them compare the Results and Conclusions obtained by the Method of Fluxions, with the like Conclusions obtained by other undisputed Principles or Methods of Calculation; and if these Results continually agree, then it is a convincing Proof (at least à posteriori*) that the Principles from whence they are deduced must be equally true. But if any Person that plainly discovers himself unacquainted with mathematical Principles, shall, out of his Aversion to these Sciences, cavil and dispute against the Principles here laid down, and which he understands nothing of; and endeavour to put the Issue on such a footing, as neither himself nor any Body else can understand any Thing about it, by running the Account of it into the dark: I think it can be of no Consequence*

ſequence at all to trouble a Man's ſelf with this Sort of Anti-Mathematicians. *Ignorance, Malice, or Prejudice can deſerve no Notice.*

And thus much I choos'd to ſay here by way of Preface, rather than in the Book itſelf, (which I would not encumber with needleſs Diſputes) concerning the Nature and Principles of Fluxions; a Thing eaſy enough to be underſtood, and rendered difficult, more by the intricate Diſputes that have been dragged into it by the Enemies of Science, than from the Nature of the Thing itſelf. The divine Newton *(whoſe Works will laſt as long as the Sun and Moon) clearly ſaw that this Matter did not at all require to be built upon any metaphyſical Speculations; he, by expreſſing the ſimple Moments by* general *Characters, did thence derive, by infinite Series, the Moments of compound Quantities; from whence he gets the Proportion of the Fluxions for any indetermined Values of theſe Moments, from which* general Proportion *he at laſt gains their Proportion for that* particular *Caſe where the Moments firſt begin, or at laſt vaniſh into nothing. And thus he has given a Demonſtration extremely eaſy, and compleat in itſelf.*

If Arts and Sciences of many hundred Years ſtanding receive daily Improvements and Additions, it cannot be ſuppoſed that this moſt ſublime Art of all, found out but Yeſterday can be arrived at Perfection all on a ſudden. If this Art be ſo exceedingly uſeful and valuable, it certainly deſerves the Pains and Attention of the learned Mathematicians. And the World muſt expect that the Beauty of this Method will excite them to lend all their Aſſiſtance towards the Advancement of ſo noble a Branch of Learning, whether it be in improving the Theory, or facilitating the Practice. Therefore I hope I ſhall be excuſed at leaſt, if, among others, I endeavour to contribute a little towards this great End.

The following Book is divided into three Sections: In the firſt *are laid down ſuch univerſal Propoſitions, as are the Foundation of all that Doctrine. The* ſecond *Section applies theſe Principles to the Solution of the moſt general*

ral Problems, or those of most frequent Use in the Mathematicks: And here many of the Problems so often done by others are resolved by Methods entirely new, and, I think, more simple; and therefore will by many Persons be more easily apprehended in this Form. And because the Resolution of physical Problems has been little touched on by others, I have added the third *Section, where you have the Investigation of some of the chief physical Problems in the Phænomena of Nature. And in this Section, it may perhaps please a Reader of the Principia (the greatest human Production that ever appeared in the World) to see many of the Author's subtle Problems resolved by his own Analysis.*

It is not in the least pretended that all Things here treated of are new; for I have collected many Things which I thought material for forming this Doctrine *into a regular System; and what was wanting I endeavoured to supply as well as I could; not that I take it to be perfect: For there are many* Desiderata *still wanting to compleat the Science. Because the Method of finding Fluents by the Tables is exceeding compendious and useful, and has yet been but very slightly passed over by the Writers on this Subject: I have been at the Pains (which was not a little) to compose a new Table, whose Use will appear upon trial to be far more easy and intelligible than any extant; and no less extensive. And for the Explanation and Use of it, I have given a vast Variety of Examples throughout the whole Book. Yet I have not omitted the most general known Rules for finding the Fluents by infinite Series; and have inserted the general Forms of them in the Table. In the Use of which Table there is not the least Difficulty, there being nothing required but a bare Substitution of Quantities. But as to the Resolution of Problems by infinite Series, I have been more sparing of that, because it has been well prosecuted by others. I am not ignorant, that (by the Method of Transmutation of Fluxions) this Table might have been further extended, and other more compounded Forms might have been inserted. But, considering how*

ſeldom theſe come in Uſe, I thought it needleſs to carry it any further. I have all along, in my Calculations, uſed diſtinct Characters for the Fluxions and Moments, ſince they ought not to be confounded together. And in moſt Problems (except ſuch where the Reaſoning is ſo obvious as not to need it) I have uſed both of them; beginning firſt with the Moments, and ſubſtituting at laſt the Fluxions for the evaneſcent Moments which are proportional thereto.

In all theſe Things I have deſignedly been very ſhort. For the general Rules and Methods of Operation being laid down in as few Words as poſſible, the Examples will explain their Meaning and Uſe. In the mathematical Sciences I have taken general Methods to be beſt; and they that deal in the detail of Things, and ſpin them out to an unneceſſary Length, making thereby a pompous ſhow of Words only, do certainly miſpend the Time of their Readers: Since one great End to be aimed at in the Sciences, is to abridge and reduce them to the moſt general and conciſe Rules.

As I am not conſcious of any Faults I have committed in this Treatiſe, ſo I hope they are but few. But in ſuch a vaſt Variety of Things of the moſt intricate Nature, it is hardly poſſible but ſome will eſcape. Therefore I muſt beg of the courteous aud good natur'd Reader (for whom alone it was written) that he will rather kindly inform me of my Errors and Defects, than cenſure and condemn my Work. For as Truth is what I ſeek, I ſhall with Pleaſure retract or correct any Thing I have written, when it appears inconſiſtent with that or the Reaſon of Things.

Laſtly let me acquaint the Reader, that it is indiſpenſably required, that he perfectly underſtand Arithmetic, Geometry, and Algebra in all their Parts and Improvements, the Methods of Series, Doctrine of Proportions, Nature of Logarithms, Mechanics, and Laws of Motion, &c. *all which are to be learned from theſe particular Sciences to which they belong. For I am clearly perſuaded that it is the beſt Method to treat every Science*

distinct and entire by itself, without the Mixture or Interposition of any Thing foreign to the Subject. And therefore in this Treatise I have delivered nothing but the pure Doctrine of Fluxions *alone. It would be but lost Labour for any Person unacquainted with these* Precognita, *to spend any Time in reading this Book; or indeed to attempt to read any such like Treatise with any tolerable Judgment. The Consequence would be, that either the Author or the Difficulty of the Subject must be blamed, as is always the Case; but never the Reader. But then if he comes thus prepared, this will make every Thing easy and pleasant, and he will then find few Difficulties here, but what he will easily surmount. All which I submit to the Perusal of the candid and judicious Reader.*

W. Emerson.

THE

THE DOCTRINE OF FLUXIONS.

Postulatum.

That any Quantity may be supposed to be generated by continual Increase.

HERE Quantities are consider'd, not as composed of an infinite Number of constituent Parts, but as described by a continued Motion. Thus a Line is described by the Motion of a Point; and a Rectangle may be conceiv'd to be generated by the Motion of one Side along the other; and Time proceeds by a regular Flux. And all other Quantities may (by Analogy) be conceived to be generated after the same Manner.

Definitions.

Definition I.

Quantities generated by a continual Increase are called *Fluents* or *Flowing Quantities*. Those Quantities that always retain the same Value are called *given*, *constant*, *standing*, or *invariable Quantities*; and those

that are continually changing their Value are called *variable* or *indetermin'd Quantities.*

Thus in a Circle the Diameter is a conſtant Quantity; and the verſed Sine and correſpondent Sine are variable Quantities: Alſo in a Parabola, the Latus Rectum is a conſtant Quantity, and the Abſciſa and correſpondent Ordinate are variable Quantities.

Def. II.

The Velocity of the Increaſe of any generated Quantity, or the Degree of Quickneſs (or Slowneſs) wherewith the new Parts of it continually ariſe, is called its *Fluxion.*

Thus when a Line is generated by the Motion of a Point, the Line itſelf is the Fluent, and the Velocity of the moving Point is ſtrictly its Fluxion. But Velocity is never properly aſcribed to any Thing but local Motion, and is uſed in this Definition, rather to *deſcribe* what is meant by the Word Fluxion, than to *define* it. Velocity is the ſame in a *particular* Senſe in Relation to the Space deſcribed, as Fluxion is in a *general* Senſe in Relation to the Fluent generated thereby. Velocity is allow'd by all to be a ſimple Idea, and ſo is Fluxion too. When a Man conſiders the Generation of ſeveral Quantities, after this Manner, he will find ſome to increaſe faſter, others ſlower; and conſequently that there are comparative Velocities (or Fluxions) of Increaſe during their Generation: And thus he will by Degrees get the Idea of a Fluxion; but without ſuch attentive Conſideration, he will never be the wiſer for all the Words in the World.

Def. III.

The indefinitely ſmall Portions of the Fluent which are generated in any indefinitely ſmall Portions of Time are called *Moments* or *Increments.* Or if the Fluent decreaſes, the Portions continually deſtroy'd are called *Decrements.*

Theſe

These Moments are the immediate Effects of the Fluxions, and are those Quantities by the continual Accession of which the Fluent increases and grows bigger and bigger: That is, any Moment consider'd alone is the adequate Effect of some single determinate Fluxion which is (consider'd as) its generating Cause. Therefore the Moments and Fluxions ought not to be confounded together, since the Moments (being generated by the Fluxions) are as different from the Fluxions, as any Effect is different from its Cause.

Def. IV.

The *Velocity*, *Variation*, or *Quickness* of *Increase* (or *Decrease*) of any Fluxion is called the *second Fluxion*; likewise the *Variation* or Quickness of *Increase* of the second Fluxion is called the *third Fluxion*, &c.

As in the Generation of any Fluent, the different Parts of it may be generated faster or slower, that is, its Fluxion at different Times may be unequal; so there must be Degrees of Variation by which it is continually changing, that is, it must have a second Fluxion. And in like Manner this second Fluxion may also be continually variable, and therefore must have a certain Degree of Variation in every Point, or a third Fluxion. And so on.

Def. V.

Contemporary Fluents are those which are supposed to be generated together or in equal Times; or which begin together and end together.

Def. VI.

In any Fluxionary Equation, a *Quantity* of the *first Order* is that which has only one first Fluxion in it; a *Quantity* of the *second Order* has either one second Fluxion or two first Fluxions: *Quantities* of the *third Order*, are third Fluxions, product of three first Fluxions, product of a first and second Fluxion, &c.

NOTATION.

1. The firſt Letters of the Alphabet, a, b, c, *&c.* are generally put for ſtanding Quantities; and the laſt, x, y, z, *&c.* for variable or flowing Quantities.

2. If x or y be put for any fluent or variable Quantity, then the ſame Letter with a Point over it $\dot{x}$, $\dot{y}$, denotes the Fluxion of x or y reſpectively; and the ſame Letters twice pointed $\ddot{x}$ and $\ddot{y}$, are the Fluxions of $\dot{x}$ and $\dot{y}$, or the ſecond Fluxions of x and y: thus $\dddot{x}$ and $\dddot{y}$ are the third Fluxions of x and y: Likewiſe $\ddddot{x}$ and $\ddddot{y}$ denote the fourth Fluxions of x and y, *&c.* alſo the Fluxion of a or b is o.

3. Again, $\acute{x}$ denotes the Moment or Increment of x, and $\acute{y}$ the Moment or Increment of y; likewiſe $\acute{\acute{x}}$ and $\acute{\acute{y}}$ denote the Moments of the Moments, or the ſecond Moments of x and y, *&c.*

4. To the common Algebraic Characters already receiv'd I add this $\propto$, which ſignifies a general Proportion; thus, $A \propto \frac{BC}{D}$, ſignifies that A is in a conſtant Ratio to $\frac{BC}{D}$; that is (if a, b, c, d be other Values of theſe Quantities) $A : \frac{BC}{D} :: a : \frac{bc}{d}$; and thus every general Proportion is to be underſtood.

AXIOM.

Quantities, which in any finite Time continually converge to Equality, and *before* the End of that Time, approach *nearer* to one another than by any given Difference, do at laſt become equal.

If any ſhould think this not clear enough to paſs for an Axiom, he may conſider it thus; let D be their ultimate Difference, therefore they cannot approach nearer to Equality, than by that given Difference D, contrary to the Hypotheſis; which is abſurd in all Caſes except when D is nothing.

SECT.

SECT. I.
The fundamental Principles and Operations of FLUXIONS.

PROP. I.

The Fluxion of any Fluent or generated Quantity is equal to the Sum of the Fluxions of all the Roots or Sides, each multiply'd continually by the Index of its Power, and by the given Fluent divided by the ſaid Root or Side.

DEMONSTRATION.

1. LET the Fluent be $bx^m y^n$; now its Fluxion muſt be a Thing real and determinate in itſelf, otherwiſe we are ſeeking that which has no Exiſtence. By the Notation, $\dot{x}$ and $\dot{y}$ are the Fluxions of x and y; and will produce Effects, that is, will generate Moments proportional to themſelves whilſt they retain their Values, which may therefore be expreſs'd by $o\dot{x}$, $o\dot{y}$.

2. Now by the Poſtulatum, theſe Moments will increaſe the Quantities x, y, which therefore will become $x + o\dot{x}$, and $y + o\dot{y}$.

3. Therefore the Fluent $bx^m y^n$ will now become $b \times \overline{x + o\dot{x}}^m \times \overline{y + o\dot{y}}^n = b \times : x^m + mx^{m-1} o\dot{x} + px^{m-2} o^2 \dot{x}^2$, &c. $\times : y^n + ny^{n-1} o\dot{y} + qy^{n-2} o^2 \dot{y}^2$, &c. where p, q are given Quantities.

4. The laſt Quantity being actually multiply'd, and $bx^m y^n$ ſubſtracted from it; we ſhall have $bmx^{m-1} y^n o\dot{x}$ $+ bnx^m y^{n-1} o\dot{y}$

$+bnx^{m}y^{n-1}o\dot{y}+bpx^{m-1}y^{n}o^{2}\dot{x}^{2}+bqx^{m}y^{n-2}o^{2}\dot{y}^{2}+bmnx^{m-1}y^{n-1}o^{2}\dot{y}\dot{x}+$, &c. for the Moment of $bx^{m}y^{n}$.

5. Therefore the Moment last found being divided by the indefinite Quantity o, will give the Fluxion of $bx^{m}y^{n}$ equal to $bmx^{m-1}y^{n}\dot{x}+bnx^{m}y^{n-1}\dot{y}+bpx^{m-1}y^{n}o\dot{x}^{2}+bqx^{m}y^{n-2}o\dot{y}^{2}+bmnx^{m-1}y^{n-1}o\dot{y}\dot{x}+$, &c. and this is the Fluxion (or Velocity) wherewith the foregoing Moment is (or may be) uniformly generated.

6. But since the (Velocity or) Fluxion is required wherewith that Moment first arises, in this Case the Moments $o\dot{x}$ and $o\dot{y}$ will also be just arising and therefore nothing, and consequently o will be nothing, and therefore all the Terms wherein it is found will be nothing.

7. Therefore the Fluxion of $bx^{m}y^{n}$ at that Moment of Time is accurately $bmx^{m-1}y^{n}\dot{x}+bnx^{m}y^{n-1}\dot{y}$. Q. E. D.

Otherwise thus.

1. Let $\acute{x}$ and $\acute{y}$ be very small Increments uniformly generated by $\dot{x}$ and $\dot{y}$, in a very small Time.

2. The Quantities x, y, $bx^{m}y^{n}$ then at the End of that Time are become $x+\acute{x}$, $y+\acute{y}$, and $b\times\overline{x+\acute{x}}\big|^{m}\times\overline{y+\acute{y}}\big|^{n}$; and the Increments generated in that Time are $\acute{x}$, $\acute{y}$ and $bmx^{m-1}y^{n}\acute{x}+bnx^{m}y^{n-1}\acute{y}+bpx^{m-2}y^{n}\acute{x}^{2}+bqx^{m}y^{n-2}\acute{y}^{2}+bmnx^{m-1}y^{n-1}\acute{x}\acute{y}+$, &c. $=$ M by Substitution, where note p, q are given Quantities.

3. Now since $\dot{x}:\dot{y}::\acute{x}:\acute{y}=\frac{\dot{y}\acute{x}}{\dot{x}}$, because the Effects of like Causes are proportional; therefore ex-

punge $\acute{y}$ out of the Value of M, and we ſhall have the Increment of x to the Increment of $bx^m y^n$ as $\acute{x}$ to M or as 1 to $\frac{M}{\acute{x}}$ that is, as $\dot{x}$ to $bmx^{m-1} y^n \dot{x} + bnx^m y^{n-1} \dot{y} + bpx^{m-1} y^n \acute{x}\dot{x} + bmnx^{m-1} y^{n-1} \acute{x}\dot{y} + bqx^m y^{n-2} \frac{\acute{x}^2 \dot{y}^2}{\dot{x}^2} +$, *&c.* $= P + Q\acute{x}$, *&c.* by Subſtitution, (putting P for the two firſt Terms, and $Q\acute{x}$, *&c.* for the reſt).

4. But $\dot{x}$ is the Fluxion of x, therefore $P + Q\acute{x}$ is the Fluxion of $bx^m y^n$. But ſince this laſt Fluxion $P + Q\acute{x}$ is not that with which the Increment begins to be generated, which we ſeek: It is evident that by diminiſhing the Time, or which is the ſame thing, diminiſhing $\acute{x}$, the Fluxion $P + Q\acute{x}$ continually converges to that Fluxion (or Velocity) wherewith the Increment firſt ariſes, and before $\acute{x}$ be diminiſh'd to nothing, is nearer to it than by any given Difference; and therefore by the Axiom when $\acute{x}$ is nothing, and conſequently $Q\acute{x}$ nothing, then P or $bmx^{m-1} y^n \dot{x} + bnx^m y^{n-1} \dot{y}$ will be the Fluxion of $bx^m y^n$. In like Manner it may be demonſtrated for any other Quantity. Q. E. D.

Cor. 1. A Fluent can have but one Fluxion: Thus the Fluxion of x can only be $\dot{x}$; of ax^2, $2ax\dot{x}$, *&c.*

Cor. 2. When the Fluxion of any Quantity is the ſame with a propoſed Fluxion, then that Quantity is its Fluent.

Scholium.

As clear and evident as this Propoſition is, yet it has been cenſur'd as falſe and erroneous; though the Perſons that object againſt its Truth were never able

to

to tell us what the Error is, nor whether the Fluxion of any Quantity is greater or leſſer than is aſſign'd by this Propoſition.

For a further Illuſtration, let us ſuppoſe that x and y are uniformly diminiſh'd by the very ſmall Increments $\acute{x}$ and $\acute{y}$. Then by reaſoning as before, we ſhall get the Fluxion (or Velocity) of the contemporary Moment of $bx^m y^n = bmx^{m-1}y^n\dot{x} + bnx^m y^{n-1}\dot{y} - bpx^{m-1}y^n\acute{x}\dot{x} - bmnx^{m-1}y^{n-1}\acute{x}\dot{y} - \frac{bqx^m y^{n-2}\acute{x}^2\dot{y}^2}{\dot{x}^2}$ &c. $=$ P$-$ Q$\acute{x}$, &c. Now by continually diminiſhing $\acute{x}$, it is manifeſt that P$-$ Q$\acute{x}$, &c. will differ from the Fluxion the Moment firſt ariſes with, by a Quantity leſs than any aſſignable, before $\acute{x}$ be reduced to nothing.

Now when we ſee that whilſt $x + \acute{x}$ converges to x, that at the ſame time P$+$Q$\acute{x}$, &c. converges to that Fluxion the Moment ariſes with, and differs in *Exceſs* from it by leſs than any given Difference, and ſtill differs the leſs, the leſs $\acute{x}$ is taken; and when $\acute{x}$ is infinitely ſmall, that Difference is infinitely ſmall in *Exceſs*: And ſeeing alſo that whilſt $x - \acute{x}$ converges towards x, likewiſe P$-$ Q$\acute{x}$, &c. approaches to the Fluxion the Moment vaniſhes with, (or the ſucceeding Moment begins with, being the ſame the other converg'd to) and differs in *Defect* from it, by leſs than any given Difference, and ſtill differs the leſs as $\acute{x}$ is leſs, and when $\acute{x}$ is infinitely diminiſh'd, that Difference is infinitely ſmall in *Defect*. If any one after all this ſhould contend, that when $\acute{x}$ is quite vaniſh'd

and become nothing, that the Fluxion of bx^my^n is not accurately P or $bmx^{m-1}y^n\dot{x} + bnx^ny^{n-1}\dot{y}$; I think no Man ought to give himself any Concern at all for such an Adversary, or take any further Pains for his Conviction.

Prop. II.

If two Fluents or variable Quantities be equal to each other, or in a given Ratio; their Fluxions will be equal, or in the same given Ratio.

And if two fluxionary Quantities are equal or in a given Ratio; their contemporary Fluents will be equal or in the same given Ratio.

For since the Quantities always continue equal, or in a given Ratio; the continually arising Increments, and therefore the Fluxions proportional thereto, will necessarily be equal or in the *same* given Ratio. And *vice versâ*, the Fluxions or generating Causes being always equal or in a given Ratio; their Effects or contemporary Fluents will therefore be equal, or in the same given Ratio. Q. E. D.

Cor. 1. Two fluxionary Quantities may be equal, and their simple Fluents unequal. For (by this Prop.) only the contemporary Fluents can be equal. And therefore

Cor. 2. A Fluxion may have an infinite Number of Fluents: thus the Fluent of $\dot{x}$ is x, $a+x$, $b+x$, $c+x$, $x-a$, $x-b$, $x-c$, &c.

Cor. 3. If any Fluxion be equal (or nearly equal) to some other Fluxion, in some particular Case: then the one may be substituted for the other in any fluxionary Equation, and their Fluents will be equal in that particular Case, (or nearly so.)

Cor. 4. If there be any Relation whatsoever between the Moments of several Quantities in their nascent or evanescent State; the same Relation there is also between the Fluxions of these Quantities.

COR. 5. And therefore in any Equation between the Moments consider'd as arising, the Fluxions may be substituted in their Room: and the contrary.

PROP. III.

Given an Equation containing the Relation of the flowing Quantities; to determine the Relation of their Fluxions.

SOLUTION.

1. If any Term contains only one variable Quantity; multiply its Index, the Fluxion of the Root, its Power whose Index is lessen'd by 1, and the Coefficient of the Term continually, for the Fluxion of that Term.

2. If any Term involves several flowing Quantities; multiply separately the Fluxion of every variable Quantity (or its Power) by all the other Quantities in the Term; the Sum of all these Products is the Fluxion of that Term.

3. Repeat the same Operation for all the Terms in the Equation; and the Sum of all gives an Equation containing the Fluxions required.

4. In exponential Equations, or those whose Exponents are variable; let X, Y, Z, &c. be the hyperbolic Logarithms of x, y, z, &c. then multiply the Index of any Quantity into the Logarithm of that Quantity, and you'll have a logarithmic Equation, whose Fluxion is to be found by the foregoing Rules. Then expunge the logarithmic Fluxions $\dot{X}$, $\dot{Y}$, or $\dot{Z}$, by substituting their Equals $\frac{\dot{x}}{x}$, $\frac{\dot{y}}{y}$, or $\frac{\dot{z}}{z}$; the Reason of which will appear hereafter by Prob. II. Sect. II.

5. Sometimes it may be convenient to divide the Equation given, by some of the indetermin'd Quantities contain'd in most of the Terms; or to substitute single Letters for compound Quantities.

EXAMPLE 1.

Let $y = a + x - z - b - dx$; then the Fluxion is $\dot{y} = \dot{x} - \dot{z} - d\dot{x}$.

Ex. 2.

Let $x = ay^3$; then the Fluxion of this Equation is $\dot{x} = 3ay^2\dot{y}$.

Ex. 3.

Let $y = ax^2$, the Fluxion is $\dot{y} = 2ax\dot{x}$. Or in particular Numbers, let $a = 10$, $\dot{x} = 1$, then $\dot{y} = 20x$. Now if $x = 1$, then $\dot{y} = 20$. If $x = 2, 3, 4,$ &c. then $\dot{y} = 40, 60, 80,$ &c.

Ex. 4.

Let $z = 120bx^n$, the Fluxion is $\dot{z} = 120nbx^{n-1}\dot{x}$.

Ex. 5.

Let $\sqrt{aa - xx} = y$, that is $\overline{aa - xx}^{\frac{1}{2}} = y$; its Fluxion is $\frac{1}{2} \times -2x\dot{x} \times \overline{aa - xx}^{-\frac{1}{2}} = \dot{y}$, or $\frac{-x\dot{x}}{\sqrt{aa - xx}} = \dot{y}$. In Numbers thus, let $a = 10$, $\dot{x} = 1$, then $\dot{y} = \frac{-x}{\sqrt{100 - xx}}$, and if $x = 0$, then $\dot{y} = \frac{-0}{10} = 0$. If $x = 3$, $\dot{y} = \frac{-3}{\sqrt{91}}$. If $x = 6$, then $\dot{y} = \frac{-3}{4}$. If $x = 10$, $\dot{y} = \frac{-10}{0} = -$ infinity; and so of others.

Ex. 6.

Let $v = bx^3y^2$, the Fluxion is $\dot{v} = 3by^2x^2\dot{x} + 2bx^3y\dot{y}$.

Ex. 7.

Let $\frac{x}{y} = v$, that is $x^1y^{-1} = v$; the Fluxion is $y^{-1}\dot{x} - 1 \times x^1y^{-2}\dot{y} = \dot{v}$, or $\frac{\dot{x}}{y} - \frac{x\dot{y}}{y^2} = \dot{v}$, that is $\frac{y\dot{x} - x\dot{y}}{yy} = \dot{v}$.

Ex. 8.

Let $x^3 - ax^2 + axy - y^3 = 0$. Its Fluxion will be $3x^2\dot{x} - 2ax\dot{x} + ax\dot{y} + ay\dot{x} - 3y^2\dot{y} = 0$.

Ex. 9.

Let this Equation be given $y^2 - a^2 = x\sqrt{aa - xx}$; its Fluxion is $2y\dot{y} = \dot{x}\sqrt{aa - xx} - \dfrac{x^2\dot{x}}{\sqrt{aa - xx}}$, or $2y\dot{y} = \dfrac{aa\dot{x} - 2x^2\dot{x}}{\sqrt{aa - xx}}$.

Ex. 10.

Let $2y^3 + x^2y - 2cyz = z^3 - 3yz^2$ be given; then its Fluxion is $6y^2\dot{y} + 2yx\dot{x} + x^2\dot{y} - 2cy\dot{z} - 2cz\dot{y} = 3z^2\dot{z} - 3z^2\dot{y} - 6yz\dot{z}$.

Or thus, divide the given Equation by y, and you'll have $2y^2 + x^2 - 2cz = \dfrac{z^3}{y} - 3z^2$, whose Fluxion is $4y\dot{y} + 2x\dot{x} - 2c\dot{z} = \dfrac{3z^2\dot{z}}{y} - \dfrac{z^3\dot{y}}{y^2} - 6z\dot{z}$; or $4y^3\dot{y} + 2xy^2\dot{x} - 2cy^2\dot{z} = 3yz^2\dot{z} - z^3\dot{y} - 6y^2z\dot{z}$.

Ex. 11.

$x^3 - ay^2 + \dfrac{by^3}{a+y} - xx\sqrt{ay + xx} = 0$, the Fluxion is $3x^2\dot{x} - 2ay\dot{y} + \dfrac{3by^2\dot{y}}{a+y} - \dfrac{by^3\dot{y}}{\overline{a+y}|^2} - 2x\dot{x}\sqrt{ay + xx}$ $\dfrac{- ax^2\dot{y} - 2x^3\dot{x}}{2\sqrt{ay + xx}} = 0$. Or $3x^2\dot{x} - 2ay\dot{y} + \dfrac{3aby^2\dot{y} + 2by^3\dot{y}}{\overline{a+y}|^2} - \dfrac{4ayx\dot{x} + 6x^3\dot{x} + ax^2\dot{y}}{2\sqrt{ay + xx}} = 0$.

Ex. 12.

Let $\sqrt{ax + \sqrt{aa - xx}} = v$, then will $ax + \sqrt{ax - xx} = vv$, whose Fluxion is $a\dot{x} - \dfrac{x\dot{x}}{\sqrt{aa - xx}} = 2v\dot{v}$,

$2v\dot v$, whence $\dot v = \frac{a\dot x}{2v} - \frac{x\dot x}{2v\sqrt{aa - xx}}$, or $\dot v = \frac{a\dot x - \frac{x\dot x}{\sqrt{aa-xx}}}{2\sqrt{ax+\sqrt{aa-xx}}}$.

Ex. 13.

Let $z\dot y - a\dot x = 0$; its Fluxion will be $\dot z\dot y + z\ddot y - a\ddot x = 0$. Or thus, let $s = \dot y$, $t = \dot x$, then $\dot s = \ddot y$, and $\dot t = \ddot x$; and the Equation $z\dot y - a\dot x = zs - at = 0$; whose Fluxion $z\dot s + s\dot z - a\dot t = 0$, that is (restoring the Values of s, $\dot s$, $\dot t$) $z\ddot y + \dot y\dot z - a\ddot x = 0$, the same as before.

Ex. 14.

Suppose $2x\dot x\dot z = ay\ddot y$, and let $\dot x$ be a given or constant Quantity, then its Fluxion is $2\dot x^2\dot z + 2x\dot x\ddot z = a\dot y\ddot y + ay\dddot y$.

Or thus, let $\dot x = b$, $\dot z = t$, $\ddot y = v$; then the given Equation becomes $2bxt = ayv$, whose Fluxion is $2bt\dot x + 2bx\dot t = av\dot y + ay\dot v$. And restoring $\dot z$, $\ddot z$, $\dot x$, $\ddot y$, $\dddot y$ for t, $\dot t$, b, v, $\dot v$; you'll have $2\dot z\dot x^2 + 2x\dot x\ddot z = a\dot y\ddot y + ay\dddot y$, the Fluxion required.

Ex. 15.

Let $\frac{az\ddot x}{\dot x^2} + \frac{y\dot y}{a} - \dot y = 0$, and suppose $\dot y$ constant; then its Fluxion is $\frac{a\dot z\ddot x}{\dot x^2} + \frac{az\dddot x}{\dot x^2} - \frac{2az\ddot x^2}{\dot x^3} + \frac{\dot y^2}{a} = 0$.

Ex. 16.

Suppose $2a^3y\ddot y - x^2\dot z^2 - x^2\dot z\dot x = a^2\dot y^2 - \frac{a^3\dot y^2\ddot y}{\dot z^2}$, and $\dot z$ invariable, its Fluxion will be $2a^3\dot y\ddot y + 2a^3y\dddot y - 2x\dot x\dot z^2$

$$2x\dot{x}\dot{z}^2 - 2x\dot{x}^2\dot{z} - x^2\dot{z}\ddot{x} = 2a^2\dot{y}\ddot{y} - \frac{2a^3\dot{y}\ddot{y}^2}{\dot{z}^2} - \frac{a^3\dot{y}^2\dddot{y}}{\dot{z}^2}.$$

Ex. 17.

Let $Y^3 = x$, where Y is the hyperbolic Log. of y; then its Fluxion is $3Y^2\dot{Y} = \dot{x}$; but $\dot{Y} = \frac{\dot{y}}{y}$, therefore $\dot{x} = \frac{3Y^2\dot{y}}{y}$.

Ex. 18.

Suppose $y^x = z$, then will $xY = Z$ (where Z, Y are the hyp. Log. of z, y) the Fluxion of this Equation is $x\dot{Y} + Y\dot{x} = \dot{Z}$; but $\dot{Y} = \frac{\dot{y}}{y}$, and $\dot{Z} = \frac{\dot{z}}{z}$, whence $\frac{x\dot{y}}{y} + Y\dot{x} = \frac{\dot{z}}{z}$; therefore $\dot{z} = \left(\frac{xz\dot{y}}{y} + Yz\dot{x} =\right) xy^{x-1}\dot{y} + Yy^x\dot{x}$.

Ex. 19.

Let $y^{x^v} = z$; then $x^vY = Z$, in Fluxions $x^v\dot{Y} + Y \times$ Fluxion of $x^v = \dot{Z}$; but (by Ex. 18.) Fluxion of $x^v = vx^{v-1}\dot{x} + Xx^v\dot{v}$; and $\dot{Y} = \frac{\dot{y}}{y}$, also $\dot{Z} = \frac{\dot{z}}{z}$; therefore $\frac{x^v\dot{y}}{y} + Yvx^{v-1}\dot{x} + YXx^v\dot{v} = \frac{\dot{z}}{z} = \frac{\dot{z}}{y^{x^v}}$: Whence $\dot{z} = x^vy^{x^v-1}\dot{y} + vYy^{x^v}x^{v-1}\dot{x} + x^vYXy^{x^v}\dot{v}$.

PROP. IV.

Let the Fluent of $\overline{e+fz^n}^m z^p\dot{z} = A$.

Fluent of $\overline{e+fz^n}^m z^{p+n}\dot{z} = B$.

Fluent of $\overline{e+fz^n}^{m+1} z^p\dot{z} = P$.

Then I say,

I. $\overline{p+1}.P + \overline{m+1}.nfB = z^{p+1}\,\overline{e+fz^n}^{m+1}$.

II. $P = eA + fB$.

DE-

DEMONSTRATION.

For I. $\overline{p+1}.\dot{P}+\overline{m+1}.nf\dot{B}=\overline{p+1}.z^p\dot{z}.\overline{e+fz^n}^{m+1}$
$+\overline{m+1}.nfz^{p+n}\dot{z}.\overline{e+fz^n}^m$
whose Fluent, by Cor. 2. Prop. I. is

$\overline{p+1}.P+\overline{m+1}.nfB = z^{p+1}\overline{e+fz^n}^{m+1}.$

II. $\overline{e+fz^n}^{m+1}z^p\dot{z}=\overline{e+fz^n}\times\overline{e+fz^n}^m z^p\dot{z}=$
$ez^p\dot{z}.\overline{e+fz^n}^m+fz^{p+n}\dot{z}.\overline{e+fz^n}^m$. That is
$\dot{P}=e\dot{A}+f\dot{B}$, and the Fluent is $P=eA+fB$.
Q. E. D.

COR. 1. Hence if any one of the Fluents A, B, P, be given, the other two will be found. Therefore,

COR. 2. $B=\dfrac{z^{p+1}.\overline{e+fz^n}^{m+1}-\overline{p+1}.eA}{\overline{p+1+mn+n}.f}.$

COR. 3. $A=\dfrac{z^{p+1}\overline{e+fz^n}^{m+1}-\overline{p+1+mn+n}.fB}{\overline{p+1}.e}.$

COR. 4. $P=\dfrac{\overline{m+1}.neA+z^{p+1}\overline{e+fz^n}^{m+1}}{p+1+mn+n}.$

COR. 5. $A=\dfrac{\overline{p+1+mn+n}.P-z^{p+1}\overline{e+fz^n}^{m+1}}{\overline{m+1}.ne}.$

PROP. V.

Let $e+fz^n+gz^{2n}+hz^{3n}$ *&c.* $=V$. $t=$ *Number of Terms in* V.

And the Fluent of $V^m z^p\dot{z}=A$.

Fluent of $V^m z^{p+n}\dot{z}=B$.

Fluent of $V^m z^{p+2n}\dot{z}=C$.

Fluent of $V^m z^{p+3n}\dot{z}=D$. *&c. continu'd to t Quantities.*

And Fluent of $V^{m+1}z^p\dot{z}=P$. *Then will,*

I. $\left.\begin{array}{l}\overline{p+1}.eA+\overline{p+1+mn+n}.fB+\\ \overline{p+1+2mn+2n}.gC+\overline{p+1+3mn+3n}.hD\\ +,\text{ \&c. continu'd to } t \text{ Terms}\end{array}\right\}=z^{p+1}V^{m+1}.$

II. P

II. $P = eA + fB + gC + hD +$ &c. to t Terms.

DEMONSTRATION.

Let $p + 1 = r$, $m + 1 = s$. And putting the first Equation into Fluxions, there is $re\dot{A} + \overline{r + sn}\,.\,f\dot{B} + \overline{r + 2sn}\,.\,g\dot{C} + \overline{r + 3sn}\,.\,h\dot{D}$, &c. $= rz^p\dot{z}V^{m+1} + sz^{p+1}V^m\dot{V}$; that is (restoring the Values of V, $\dot{V}$, $\dot{A}$, $\dot{B}$, &c. and dividing by $z^pV^m\dot{z}$) $re + \overline{r + sn}\,.\,fz^n + \overline{r+2sn}\,.\,gz^{2n} + \overline{r+3sn}\,.\,hz^{3n}$ &c. $= r \times \overline{e + fz^n + gz^{2n} + hz^{3n}}$ &c. $+ sz \times \overline{nfz^{n-1} + 2ngz^{2n-1} + 3nhz^{3n-1}}$ &c. where both Sides of the Equation being manifestly equal, 'tis evident the Quantities from whence they were derived, that is, the Fluents in Equation the 1st, must also be equal. After the same Manner is the second Equation very easily demonstrated. Q. E. D.

COR. Hence if $t - 1$ of the Fluents P, A, B, C, &c. be given; the rest will be found.

PROP. VI.

Put $e + fz^n = V$, $k + lz^n = Y$, *and*
the Fluent of $z^p\dot{z}V^mY^q = A$.
Fluent of $z^{p+n}\dot{z}V^mY^q = B$.
Fluent of $z^{p+2n}\dot{z}V^mY^q = C$.
Also Fluent of $z^p\dot{z}V^{m+1}Y^q = P$.
Fluent of $z^{p+n}\dot{z}V^{m+1}Y^q = Q$.

Then it will be

$$\text{I.}\quad \left.\begin{array}{l} \overline{p+1}\,.\,ekA \\ \left.\begin{array}{l} +\overline{p+1+mn+n}\,.\,fk \\ +\overline{p+1+qn+n}\,.\,el \end{array}\right\} B \\ +\overline{p+1+mn+qn+2n}\,.\,flC \end{array}\right\} = z^{p+1}V^{m+1}Y^{q+1}.$$

II. $P = eA + fB$.

III. $Q = eB + fC$.

DEMONSTRATION.

Let $p + 1 = r$, $m + 1 = s$, $q + 1 = t$, and putting the first Equation into Fluxions, we have

$rek\dot{A}$

$$rek\dot{A}\left.\begin{matrix}+\overline{r+sn}.fk\\+\overline{r+tn}.el\end{matrix}\right\}\dot{B}+\overline{r+sn+tn}.fl\,\dot{C}=$$

$$rz^p\dot{z}V^{m+1}Y^{q+1}+sz^{p+1}Y^{q+1}V^m\dot{V}+tz^{p+1}V^{m+1}Y^q\dot{Y}.$$

Then restoring the Values of $\dot{A}$, $\dot{B}$, $\dot{C}$, $\dot{V}$, $\dot{Y}$, and dividing by $z^p\dot{V}^mY^q$, and then expunging V, Y; we have $rek\left.\begin{matrix}+\overline{r+sn}.fk\\+\overline{r+tn}.el\end{matrix}\right\}z^n+\overline{r+sn+tn}.flz^{2n}$ $=r.\overline{e+fz^n}\times\overline{k+lz^n}+snfz^n\times\overline{k+lz^n}+tnlz^n\times$ $\overline{e+fz^n}$, an Equation whose two Sides (when reduced) being manifestly the same, argues the Equation from whence it was deduced to be a true Equation. And the same may be easily shown of the other two Equations. Q. E. D.

Cor. Hence if any 2 of the Fluents A, B, C, P, Q be given; the rest may be found.

PROP. VII.

Let $e+fz^n+gz^{2n}+hz^{3n}$ *&c.* $=V$.
$k+lz^n+rz^{2n}+sz^{3n}$ *&c.* $=Y$.
$t=$ *Number of Terms in* V, $\tau=$ *Number of Terms in* Y.

And the Fluent of $z^p\dot{z}V^mY^q=A$.
Fluent of $z^{p+n}\dot{z}V^mY^q=B$.
Fluent of $z^{p+2n}\dot{z}V^mY^q=C$.
Fluent of $z^{p+3n}\dot{z}V^mY^q=D$. *&c. continu'd to* $t+\tau-1$ *Quantities.*

Also Fluent of $z^p\dot{z}V^{m+1}Y^q=P$.
Fluent of $z^{p+n}\dot{z}V^{m+1}Y^q=Q$.
Fluent of $z^{p+2n}\dot{z}V^{m+1}Y^q=R$. *&c. continu'd to* τ *Quantities.*

Then I say,

$$\text{I. }\overline{p+1}.ekA\left.\begin{matrix}+\overline{p+1+qn+n}.el\\+\overline{p+1+mn+n}.fk\end{matrix}\right\}B.$$

$$\left.\begin{array}{l}+\overline{p+1+2qn+2n}.er\\ +\overline{p+1+mn+qn+2n}.fl\\ +\overline{p+1+2mn+2n}.gk\end{array}\right\}C\ \left.\begin{array}{l}+\overline{p+1+3qn+3n}.es\\ +\overline{p+1+mn+2qn+3n}.fr\\ +\overline{p+1+2mn+qn+3n}.gl\\ +\overline{p+1+3mn+3n}.hk\end{array}\right\}D$$

$+$ &c. (continu'd to $t+\tau-1$ Terms) $=$ $z^{p+1}V^{m+1}Y^{q+1}$.

II. $P = eA + fB + gC + hD$ &c. to t Terms.
III. $Q = eB + fC + gD + hE$ &c.
IV. $R = eC + fD + gE + hF$ &c.
V. $S = eD + fE + gF + hG$ &c.
&c. continued to τ Equations.

This is demonstrated like the foregoing.

COR. Hence if $t+\tau-2$ of the Fluents P, Q, R, A, B, C, &c are given, the rest will be found.

PROP. VIII.

Let $V = e + fz^n$. $Y = k + lz^n$. $X = s + tz^n$.

And the Fluent of $z^p\dot{z}V^mY^qX^r = A$.
Fluent of $z^{p+n}\dot{z}V^mY^qX^r = B$.
Fluent of $z^{p+2n}\dot{z}V^mY^qX^r = C$.
Fluent of $z^{p+3n}\dot{z}V^mY^qX^r = D$.
Also Fluent of $z^p\dot{z}V^{m+1}Y^qX^r = P$.
Fluent of $z^{p+n}\dot{z}V^{m+1}Y^qX^r = Q$.
Fluent of $z^{p+2n}\dot{z}V^{m+1}Y^qX^r = R$.

Then it will be

$$\text{I. }\overline{p+1}.eksA\ \left.\begin{array}{l}+\overline{p+1+mn+n}.fks\\ +\overline{p+1+qn+n}.els\\ +\overline{p+1+rn+n}.ekt\end{array}\right\}B$$

$$\left.\begin{array}{l}+\overline{p+1+mn+qn+2n}.fls\\ +\overline{p+1+mn+rn+2n}.fkt\\ +\overline{p+1+qn+rn+2n}.elt\end{array}\right\}C+\overline{p+1+mn+qn+rn+3n}$$

$\times fltD = z^{p+1}V^{m+1}Y^{q+1}X^{r+1}$.

II. $P = eA + fB$.
III. $Q = eB + fC$.
IV. $R = eC + fD$.

The Truth of this is ſhewn as the reſt.

Cor. Hence if (any) three of the Fluents A, B, C, D, P, Q, R be given, all the reſt will be found.

SCHOLIUM.

In the five laſt Propoſitions it muſt be obſerved that any of the Fluents A, B, C; P, Q, R, &c. vaniſhes out of the Equation when its Coefficient is o: And therefore in ſuch a Caſe that Fluent cannot be found by any of theſe Propoſitions.

If any one ſhould aſk, how theſe Propoſitions are found out? The Anſwer is, by the common Analyſis. For taking any Equation in theſe Propoſitions, or aſſuming any Equation among the Fluents A, B, C, P, Q, R, at pleaſure, and affecting them with unknown Coefficients; then putting that Equation into Fluxions, (ſubſtituting the Values of $\dot{A}$, $\dot{B}$, $\dot{P}$, &c.) and then comparing the homologous Terms, theſe Coefficients will eaſily be determin'd, if the Thing is poſſible. And thus you may find out other Propoſitions of this Kind.

PROP. IX.

To transform a given Fluxion into another more ſimple one.

RULE.

1. For any compound Quantity in the given Fluxion put a new Letter or variable Quantity, by help of which expunge the other variable Quantity and its Fluxion out of the given fluxionary Quantity; ſo will you have a new Fluxion inſtead of the former. If this be not ſimple enough, aſſume another variable Quantity inſtead of any compound Quantity contain'd in this laſt, and expunge the former aſſum'd Quantity and its Fluxion. Proceed thus till the transform'd Fluxion be as ſimple as poſſible.

2. Sometimes a compound Fluxion may be diſſolved into ſeveral other ſimpler ones, thus; aſſume two or

more such fluxionary Quantities, as you conceive will, when duly reduc'd, make up the given Fluxion, and let them be affected with indetermin'd Coefficients; then reducing them to the Form of the given Fluxion, these Coefficients will be easily determin'd by comparing the homologous Terms.

Ex. 1.

Let $\overline{e+fz^n}^m \times \overline{g+hz^n}|^s z^{rn-1}\dot{z}$ be proposed. Assume $v=e+fz^n$, then $z^n=\frac{v-e}{f}$, and $z^{rn}=\frac{\overline{v-e}|^r}{f^r}$, and $z^{rn-1}\dot{z}=\frac{\overline{v-e}|^{r-1}\dot{v}}{nf^r}$, and $g+hz^n=\frac{fg-he+hv}{f}$ $=\frac{p+hv}{f}$, putting $p=fg-eh$. Therefore the given Fluxion $\overline{e+fz^n}^m \times \overline{g+hz^n}|^s z^{rn-1}\dot{z}=\frac{\overline{v-e}|^{r-1}}{nf^{r+s}}$ $\times \overline{p+hv}|^s v^m\dot{v}$.

Ex. 2.

There is given $\frac{z^{rn-1}\dot{z}}{\overline{k+lz^n}\,.\,\overline{e+fz^n+gz^{2n}}}$; assume v $=k+lz^n$, then $z^n=\frac{v-k}{l}$, and $z^{rn}=\frac{\overline{v-k}|^r}{l^r}$, $z^{rn-1}\dot{z}=\frac{\overline{v-k}|^{r-1}}{nl^r}\dot{v}$; put $p=\sqrt{ff-4eg}$, and then

$$\frac{1}{e+fz^n+gz^{2n}}=\frac{g}{p\,.\,\frac{f-p}{2}+gz^n}-\frac{g}{p\,.\,\frac{f+p}{2}+gz^n}$$

as will be evident by reducing the two Terms to a common Denominator. Therefore, putting $a=$ $\frac{f-p}{2}l-gk$, and $b=\frac{f+p}{2}l-gk$, the given Fluxion becomes $\frac{g}{p}\times\frac{\overline{v-k}|^{r-1}v^{-1}\dot{v}}{nl^{r-1}.\overline{a+gv}}-\frac{\overline{v-k}|^{r-1}gv^{-1}\dot{v}}{npl^{r-1}.\overline{b+gv}}$.

Ex.

Ex. 3.

Let $\dfrac{\overline{e+fz^n}^m z^{\lambda n-1}\dot{z}}{\overline{g+bz^n}^{m+r}}=\dot{F}$ be given. Put $v=g+bz^n$. Then $z^n=\dfrac{v-g}{b}$, and $e+fz^n=\dfrac{fv-p}{b}$ (putting $p=fg-eb$) and $z^{\lambda n-1}\dot{z}=\dfrac{\overline{v-g}^{\lambda-1}\dot{v}}{nb^\lambda}$; whence $\dot{F}=\dfrac{\overline{fv-p}^m\times\overline{v-g}^{\lambda-1}v^{-m-r}\dot{v}}{nb^{\lambda+m}}=\dfrac{\overline{1-gv^{-1}}\big\rangle^{\lambda-1}}{nb^{m+\lambda}}\times\overline{f-pv^{-1}}\big\rangle^m v^{\lambda-r-1}\dot{v}$.

Again, let $f-pv^{-1}=y$; then $v^{\lambda-r-1}\dot{v}=\dfrac{\overline{f-y}^{r-\lambda-1}\dot{y}}{p^{r-\lambda}}$, and $1-gv^{-1}=\dfrac{gy-eb}{p}$; whence $\dot{F}=\dfrac{\overline{gy-eb}^{\lambda-1}}{nb^{m+\lambda}p^{r-1}}\times\overline{f-y}^{r-\lambda-1}y^m\dot{y}$.

Otherwise.

Since $\dfrac{\overline{1-gv^{-1}}^{\lambda-1}}{nb^{m+\lambda}}\times\overline{f-pv^{-1}}^m v^{\lambda-r-1}\dot{v}$ is (by first multiplying and then dividing, by $1-gv^{-1}$) $=\dfrac{\overline{1-gv^{-1}}\big\rangle^\lambda}{nb^{m+\lambda}}\times\overline{f-pv^{-1}}^m v^{\lambda-r-1}\dot{v}+\dfrac{g.\overline{1-gv^{-1}}\big\rangle^{\lambda-1}}{nb^{m+\lambda}}\times\overline{f-pv^{-1}}^m v^{\lambda-r-2}\dot{v}$; therefore expunging v and it becomes $\dot{F}=\dfrac{\overline{gy-eb}^\lambda}{nb^{m+\lambda}p^r}\times\overline{f-y}^{r-\lambda-1}y^m\dot{y}+\dfrac{g.\overline{gy-eb}^{\lambda-1}}{nb^{m+\lambda}p^r}\times\overline{f-y}^{r-\lambda}y^m\dot{y}$, and $y=f-pv^{-1}=\dfrac{e+fz^n}{g+bz^n}b$.

Ex. 4.

Suppose $\overline{e+fz^n+gz^{2n}}^m z^{rn-1}\dot{z}=\dot{F}$, put $e+fz^n+gz^{2n}=v$, $p=\frac{1}{4}ff-eg$, then $z^n=\dfrac{-\frac{1}{2}f+\sqrt{p+gv}}{g}$, and

and $z^{rn-1}\dot{z} = \frac{\overline{-\frac{1}{2}f + \sqrt{p+gv}}\,|^{r-1}}{g^r} \times \frac{g\dot{v}}{2n\sqrt{p+gv}}$; therefore $\dot{F} = \frac{\overline{-\frac{1}{2}f+\sqrt{p+gv}}\,|^{r-1}}{2ng^{r-1}} \times \frac{v^m\dot{v}}{\sqrt{p+gv}}$. If this be not thought simple enough, suppose again $x = \sqrt{p+gv}$, then $v = \frac{xx-p}{g}$, and $v^{m+1} = \frac{\overline{xx-p}\,|^{m+1}}{g^{m+1}}$, and $v^m\dot{v} = \frac{\overline{xx-p}\,|^m}{g^{m+1}} \times 2x\dot{x}$: therefore $\dot{F} = \frac{\overline{x-\frac{1}{2}f}\,|^{r-1}}{ng^{m+r}} \times \overline{x^2-p}\,|^m\dot{x}$. Here $x = \sqrt{p+gv} = \frac{1}{2}f + gz^n$.

Ex. 5.

To transform the Fluxion $\frac{z^{\lambda n-1}\dot{z}}{e+fz^n+gz^{2n}}$, assume $\frac{gz^{\lambda n-1}\dot{z}}{ge+gfz^n+ggz^{2n}} = \frac{Az^{\lambda n-1}\dot{z}}{B+gz^n} + \frac{Cz^{\lambda n-1}\dot{z}}{D+gz^n} =$ (by reducing to a common Denominator)

$$\frac{\begin{matrix}+AD\\+BC\end{matrix} z^{\lambda n-1}\dot{z} \begin{matrix}+Ag\\+Cg\end{matrix} z^{\lambda n+n-1}\dot{z}}{BD \begin{matrix}+Bg\\+Dg\end{matrix} z^n + ggz^{2n}}$$

; then comparing the homologous Terms, $Ag+Cg=0$, $AD+BC=g$, $BD=ge$, $Bg+Dg=fg$; whence is had $C=-A$, $D+B=f$, $D-B=\sqrt{ff-4eg}=p$: and thence $A=\frac{g}{p}$, $B=\frac{f-p}{2}$, $C=\frac{-g}{p}$, $D=\frac{f+p}{2}$. And therefore

$$\frac{z^{\lambda n-1}\dot{z}}{e+fz^n+gz^{2n}} = \frac{g}{p} \times \frac{z^{\lambda n-1}\dot{z}}{\frac{f-p}{2}+gz^n} - \frac{g}{p} \times \frac{z^{\lambda n-1}\dot{z}}{\frac{f+p}{2}+gz^n}.$$

Ex. 6.

Let $\frac{z^{\lambda n-1}\dot{z}}{e+fz^n+gz^{2n}}$ be given, where λ is half an odd Number, and $4eg$ greater than ff.

Assume

Assume $\frac{gz^{\lambda n-1}\dot{z}}{eg+fgz^n+ggz^{2n}} = \frac{Az^{\lambda n-\frac{1}{2}n-1}\dot{z}}{C-Dz^{\frac{1}{2}n}+gz^n} - \frac{Bz^{\lambda n-\frac{1}{2}n-1}\dot{z}}{C+Dz^{\frac{1}{2}n}+gz^n}$ = (by Reduction)

$$\frac{\begin{matrix}+AC\\-BC\end{matrix}z^{\lambda n-\frac{1}{2}n-1}\dot{z}\begin{matrix}+AD\\+BD\end{matrix}z^{\lambda n-1}\dot{z}\begin{matrix}+Ag\\-Bg\end{matrix}z^{\lambda n+\frac{1}{2}n-1}\dot{z}}{CC+\begin{matrix}2Cg\\-DD\end{matrix}z^n+ggz^{2n}}.$$

Now comparing the homologous Terms, we have $AC - BC = 0 = Ag - Bg$, $AD + BD = g$, $CC = eg$, $5Cg - DD = fg$. From whence we have $C = \sqrt{eg}$, $D = \sqrt{2g\sqrt{eg}-fg}$, $A = \frac{g}{2D} = B$. Whence $\frac{z^{\lambda n-1}\dot{z}}{e+fz^n+gz^{2n}} = \frac{g}{2D}\times\frac{z^{\lambda n-\frac{1}{2}n-1}\dot{z}}{\sqrt{eg}-Dz^{\frac{1}{2}n}+gz^n} - \frac{g}{2D}\times\frac{z^{\lambda n-\frac{1}{2}n-1}\dot{z}}{\sqrt{eg}+Dz^{\frac{1}{2}n}+gz^n}$: both which Terms may be transform'd into others still more simple, after the Manner of the fourth Example.

Ex. 7.

Let $\frac{z^{\lambda n-1}\dot{z}}{\overline{k+lz^n}.\overline{e+fz^n+gz^{2n}}}$ be proposed, where λ is half an odd Number, and $4eg$ greater than ff. Assume for it $\frac{Az^{\lambda n-n-1}\dot{z}+Bz^{\lambda n-1}\dot{z}}{e+fz^n+gz^{2n}}+\frac{Dz^{\lambda n-n-1}\dot{z}}{k+l^n}$, or for Brevity's sake (dividing by $z^{\lambda n-n-1}\dot{z}$) $\frac{z^n}{\overline{k+lz^n}.\overline{e+fz^n+gz^{2n}}} = \frac{A+Bz^n}{e+fz^n+gz^{2n}}+\frac{D}{k+lz^n}$ = (by Reduction)

$$\frac{\begin{matrix}+Ak\\+De\end{matrix}\begin{matrix}+Alz^n\\+Bk\\+Df\end{matrix}\begin{matrix}+Blz^{2n}\\+Dg\end{matrix}}{\overline{k+lz^n}.\overline{e+fz^n+gz^{2n}}};$$

and comparing the homologous Terms, $Ak + De = 0$, $Al + Bk + Df = 1$, $Bl + Dg = 0$, whence (putting $t = ell - fkl$

$-fkl+gkk$) we have $D=\frac{-lk}{t}$, $A=\frac{el}{t}$, $B=\frac{gl}{t}$

Whence $\frac{z^{\lambda n-1}\dot{z}}{k+lz^n.e+fz^n+gz^{2n}}=\frac{elz^{\lambda n-n-1}\dot{z}+gkz^{\lambda n-1}\dot{z}}{t.e+fz^n+gz^{2n}}$ $-\frac{lkz^{\lambda n-n-1}\dot{z}}{t.k+lz^n}$; and the first Part may be transform'd again into others more simple by the sixth Example.

PROP. X.

An Equation being given containing the Fluxions of Quantities; to find the Fluents, either in simple Terms or in a Series thereof proceeding ad infinitum.

RULES.

When the *Fluxions* are *not* of the *same Order* in all the Terms, supply the Defect by the Powers of some *given Fluxion* suppos'd to be Unity: *Fractional* and *compound Quantities* must be *reduc'd* by Multiplication, Division, &c. Radical Quantities (except such where the fluxionary Part is in a given Ratio to the Fluxion of the Root) must be *reduced* to simple Terms by *Involution*; and the Roots of *adfected Equations* must be *extracted*. This Preparation being made,

I.

1. If the Equation then can be so order'd, that *every Term* has only *one variable Quantity* and its *Fluxion:* multiply separately *each Term* by its *variable Quantity*, and then *divide* it by the *Fluxion* and its *new Index*.

2. If any Term be such a *radical Quantity* that the fluxionary Part may be divided by the Fluxion of the *Root* (or Part under the Vinculum) and that by such Division the Quotient may be a given Quantity. *Multiply* that Term by the said *Root*, and then divide by the Fluxion of the *Root* and the new *Index*, for the Fluent of that radical Quantity.

3. If

3. If any Term be divided by the *first Power* of the variable Quantity; then the Fluxion of that Term must be found by itself thus; multiply the *given Coefficient* and the Number 2.302585 into the *Logarithm* of that variable Quantity, for the *Fluent* of that Term. Or thus by Series, substitute for this *variable Quantity* the Sum or Difference of some *given Quantity* and another *variable Quantity*, and its Fluxion for the Fluxion; then this new Term being actually *divided*, the Fluent will be found as in the first Article.

4. If in one Side of an Equation there be *two Terms* containing *two variable Quantities*, each multiply'd into the Fluxion of the other; then by Art. 1. find the Fluent of either Term supposing only one Quantity variable, and this will be the Fluent of both these Terms. And if there be *three Terms* containing *three variable Quantities*, where the Fluxion of each is multiply'd into the Product of the other two; then find the Fluent of any one of these Terms (by Art. 1.) considering the other Quantities as given; and this will be the Fluent of all three.

5. Lastly, All the *Terms* being *collected* on the corresponding Sides of the Equation, will give an Equation containing the *Relation* of the variable Quantities.

EXAMPLE I.

Let $\dot{y} = d\dot{x} - \dot{x}$, then the Fluent $y = dx - x$.

Ex. 2.

Let $\dot{x} = ay^2\dot{y}$, the Fluent is $x = \frac{ay^3}{3}$.

Ex. 3.

Let $\dot{y} = \frac{-x\dot{x}}{\sqrt{aa-xx}}$, or $\dot{y} = \overline{aa-xx}^{-\frac{1}{2}} \times -x\dot{x}$; multiply by the Root $aa-xx$, and there is $\overline{aa-xx}^{\frac{1}{2}} \times -x\dot{x}$; divide this by $-2\dot{x}x \times \frac{1}{2}$ and there arises $\overline{aa-xx}^{\frac{1}{2}}$ or $\sqrt{aa-xx} = y$.

Ex. 4.

Suppose $\dot{y} = \frac{a\dot{x}}{x}$, then $y = 2 . 302585a \times \text{Log. } x$. Or thus, let $b + z = x$, and $\dot{z} = \dot{x}$, then will $\dot{y} = \frac{a\dot{z}}{b + z} = \frac{a\dot{z}}{b} - \frac{az\dot{z}}{bb} + \frac{az^2\dot{z}}{b^3} - \frac{az^3\dot{z}}{b^4} +$, &c. and the Fluent $y = \frac{az}{b} - \frac{az^2}{2b^2} + \frac{az^3}{3b^3} - \frac{az^4}{4b^4}$ &c.

Ex. 5.

Let $\dot{z} = 2ay^3x\dot{x} + 3ax^2y^2\dot{y}$, where there are the Quantities y^3 and ax^2, and one multiply'd into the other's Fluxion; the Fluent of $2ay^3x\dot{x}$ (supposing y invariable) is ay^3x^2, therefore $z = ay^3x^2$.

Ex. 6.

$\dot{v} = ax^m\dot{x}$, then $v = \frac{ax^{m+1}}{m+1}$.

Ex. 7.

Suppose $\dot{y} = \overline{a\dot{x} + x\dot{x}}\sqrt{2ax + xx}$, then $y = \frac{\overline{a\dot{x} + x\dot{x}} \times \overline{2ax + xx}^{\frac{3}{2}}}{\frac{3}{2} \times 2a\dot{x} + 2x\dot{x}} = \frac{\overline{2ax + xx}^{\frac{3}{2}}}{3}$.

Ex. 8.

Let $\dot{v} = \frac{\dot{x}}{y} - \frac{x\dot{y}}{yy}$, that is $\dot{v} = y^{-1}\dot{x} - xy^{-2}\dot{y}$, and the Fluent $v = xy^{-1} = \frac{x}{y}$.

Ex. 9.

Let $\dot{v} = y^2z^3\dot{x} + 2xz^3y\dot{y} + 3xy^2z^2\dot{z}$; then $v = xy^2z^3$.

Ex. 10.

Suppose $\dot{y} = \frac{\dot{x}}{x^3} - \frac{\dot{x}}{x^2} + \frac{a\dot{x}}{x^{\frac{1}{2}}} - x^{\frac{1}{2}}\dot{x} + x^{\frac{3}{2}}\dot{x}$, multiply the first Side by $\frac{y}{\dot{y}}$, and the last by $\frac{x}{\dot{x}}$; and there will be $y = \frac{1}{xx} - \frac{1}{x} + ax^{\frac{1}{2}} - x^{\frac{3}{2}} + x^{\frac{5}{2}}$; then divide by the Index of the Power in each Term, and the Fluent will be $y = \frac{-1}{2xx} + \frac{1}{x} + 2ax^{\frac{1}{2}} - \frac{2}{3}x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}}$.

Ex. 11.

Let the given Equation be $\dot{x} = \frac{2bbc\dot{y}}{\sqrt{ay^3}} + \frac{3y^2\dot{y}}{a+b} + \dot{y}\sqrt{by+cy}$, that is $\dot{x} = \frac{2bbc}{\sqrt{a}} \times y^{-\frac{3}{2}}\dot{y} + \frac{3}{a+b} \times y^2\dot{y} + y^{\frac{1}{2}}\dot{y}\sqrt{b+c}$; the Fluent is $x = \frac{-4bbc}{\sqrt{a}}y^{-\frac{1}{2}} + \frac{y^3}{a+b} + \frac{2}{3}y^{\frac{3}{2}}\sqrt{b+c}$.

Ex. 12.

Let $\dot{y} = x^{-\frac{1}{2}}\dot{x} - x^{\frac{1}{2}}\dot{x} + \frac{\dot{x}}{x}$; the Fluent will be $y = 2x^{\frac{1}{2}} - \frac{2}{3}x^{\frac{3}{2}} + 2.302585$ Log. x. Or thus put $a - z = x$, and $-\dot{z} = \dot{x}$, then $\frac{\dot{x}}{x} = \frac{-\dot{z}}{a-z} = -\frac{\dot{z}}{a} - \frac{z\dot{z}}{aa} - \frac{z^2\dot{z}}{a^3}$ — &c. whence $\dot{y} = x^{-\frac{1}{2}}\dot{x} - x^{\frac{1}{2}}\dot{x} - \frac{\dot{z}}{a} - \frac{z\dot{z}}{a^2} - \frac{z^2\dot{z}}{a^3}$ &c. and then $y = 2x^{\frac{1}{2}} - \frac{2}{3}x^{\frac{3}{2}} - \frac{z}{a} - \frac{z^2}{2aa} - \frac{z^3}{3a^3}$ — &c.

Ex. 13.

Let the Equation be $\dot{y} = \overline{\alpha + \beta z^n}^{\mu} z^{\pi}\dot{z}$; involve that radical Quantity, and then $\dot{y} = \alpha^{\mu}z^{\pi}\dot{z} +$

 $\mu\alpha^{\mu-1}\beta z^{n+\pi}\dot{z}$

$\mu\alpha^{\mu-1}\beta z^{n+\pi}\dot z + \mu.\frac{\mu-1}{2}\alpha^{\mu-2}\beta^2 z^{2n+\pi}\dot z +$ &c. whence the Fluent is $y = \frac{\alpha^\mu z^{\pi+1}}{\pi+1} + \frac{\mu\alpha^{\mu-1}\beta z^{n+\pi+1}}{n+\pi+1} + \mu \times \frac{\mu-1}{2} \times \frac{\alpha^{\mu-2}\beta^2 z^{2n+\pi+1}}{2n+\pi+1} +$ &c.

Ex. 14.

Suppose $\dot y = \theta e z^{\theta-1}\dot z + \begin{matrix}\theta f\\ +\mu n f\end{matrix} z^{\theta+n-1}\dot z$ into $\overline{e+fz^n}^{\mu-1}$, that is $\dot y = \theta z^{\theta-1}\dot z \times \overline{e+fz^n}^{\mu} + \mu n f z^{\theta+n-1}\dot z \times \overline{e+fz^n}^{\mu-1}$, here are the two Quantities z^θ and $\overline{e+fz^n}^{\mu}$ each multiply'd into the other's Fluxion; therefore $y = z^\theta \times \overline{e+fz^n}^{\mu}$.

Ex. 15.

Let $\dot y = \theta e z^{\theta-1}\dot z + \begin{matrix}\theta f\\ +\mu n f\end{matrix} z^{\theta+n-1}\dot z + \begin{matrix}\theta g\\ +2\mu n g\end{matrix} z^{\theta+2n-1}\dot z$ into $\overline{e+fz^n+2gz^{2n}}^{\mu-1}$, that is $\dot y = \theta z^{\theta-1}\dot z \times \overline{e+fz^n+gz^{2n}}^{\mu} + \mu f n z^{\theta+n-1}\dot z + 2\mu n g z^{\theta+2n-1}\dot z \times \overline{e+fz^n+gz^{2n}}^{\mu-1}$. Here are two Quantities z^θ and $\overline{e+fz^n+gz^{2n}}^{\mu}$ each multiply'd into the Fluxion of the other, therefore $y = z^\theta \times \overline{e+fz^n+gz^{2n}}^{\mu}$, the Fluent.

Ex. 16.

Suppose $\dot y = \theta e g z^{\theta-1}\dot z + \overline{\theta+\mu n} \times fg z^{\theta+n-1}\dot z + \begin{matrix} \\ +\overline{\theta+rn}\times he\end{matrix}$ $\overline{\theta+\mu n+rn} \times fh z^{\theta+2n-1}\dot z$ into $\overline{e+fz^n}^{\mu-1} \times \overline{g+hz^n}^{r-1}$, that is $\dot y = \theta z^{\theta-1}\dot z \times \overline{e+fz^n}^{\mu} \times \overline{g+hz^n}^{r} + \mu n f z^{\theta+n-1}\dot z \times \overline{g+hz^n}^{r} \times \overline{e+fz^n}^{\mu-1} + rnhz^{\theta+n-1}\dot z \times \overline{e+fz^n}^{\mu} \times \overline{g+hz^n}^{r-1}$. Now here are the 3 Quantities z^θ, $\overline{e+fz^n}^{\mu}$ and $\overline{g+hz^n}^{r}$; and the Fluxion of each is multiply'd into the Product of the rest; therefore the Fluent is $y = z^\theta \times \overline{e+fz^n}^{\mu} \times \overline{g+hz^n}^{r}$.

Ex.

Ex. 17.

Let the Equation be given $\dot{y}^2 = \dot{x}\dot{y} + x^2\dot{x}^2$; by extracting the Root, $\dot{y} = \frac{1}{2}\dot{x} \pm \dot{x}\sqrt{\frac{1}{4} + xx} =$ (by Involution) $\frac{1}{2}\dot{x} + \frac{1}{2}\dot{x} + x^2\dot{x} - x^4\dot{x} + 2x^6\dot{x}$ &c. or $\dot{y} = \frac{1}{2}\dot{x} - \frac{1}{2}\dot{x} - x^2\dot{x} + x^4\dot{x} - 2x^6\dot{x}$ &c. Whence $y = x + \frac{x^3}{3} - \frac{x^5}{5} + \frac{2x^7}{7}$ &c. or $y = \frac{-x^3}{3} + \frac{x^5}{5} - \frac{2x^7}{7}$ &c.

Ex. 18.

Suppose $\dot{y}^3 + ax\dot{x}^2\dot{y} + a^2\dot{x}^2\dot{y} - x^3\dot{x}^3 - 2a^3\dot{x}^3 = 0$. Extracting the Root of this Equation, $\dot{y} = a\dot{x} - \frac{x\dot{x}}{4} + \frac{x^2\dot{x}}{64a} + \frac{131x^3\dot{x}}{512a^2}$ &c. then the Fluent will be $y = ax - \frac{x^2}{8} + \frac{x^3}{192a} - \frac{131x^4}{2048a^2} +$ &c.

Ex. 19.

Suppose this Equation $\dot{z}^3 - c\dot{z}^2 - 2x^2\dot{z} - c^2\dot{z} + 2x^3 + c^3 = 0$. Put $\dot{x} = 1$, by which multiplying the deficient Terms, and then you have $\dot{z}^3 - c\dot{z}^2\dot{x} - 2x^2\dot{x}^2\dot{z} - c^2\dot{x}^2\dot{z} + 2x^3\dot{x}^3 + c^3\dot{x}^3 = 0$. And extracting the Root, which will be threefold, then $\dot{z} = c\dot{x} + x\dot{x} - \frac{x^2\dot{x}}{4c} + \frac{x^3\dot{x}}{32cc}$ &c. Or $\dot{z} = c\dot{x} - x\dot{x} + \frac{3x^2\dot{x}}{4c} + \frac{15x^3\dot{x}}{32cc}$ &c. Or lastly $\dot{z} = -c\dot{x} - \frac{x^2\dot{x}}{2c} - \frac{x^3\dot{x}}{2cc} + \frac{x^5\dot{x}}{4c^4}$ &c. Whence the Fluent will either be $z = cx + \frac{1}{2}xx - \frac{x^3}{12c} + \frac{x^4}{128cc}$ &c. Or $z = cx - \frac{1}{2}xx + \frac{x^3}{4c} + \frac{15x^4}{128cc}$ &c. Or $z = -cx - \frac{x^3}{6c} - \frac{x^4}{8cc} + \frac{x^6}{24c^4}$ &c.

Ex.

Ex. 20.

Let $\frac{a^4\dot{y}^2}{y} - 4a^3\dot{y}^2 + 4aay\dot{y}^2 + \frac{aaxx}{y}\dot{y}^2 = 4ax^2\dot{y}^2 + \frac{x^2\dot{x}^2}{y} - 4x^2y\dot{y}^2$; transpose $4ax^2\dot{y}^2 - 4x^2y\dot{y}^2$, and then multiply by y, and you have $a^4\dot{y}^2 - 4a^3y\dot{y}^2 + 4a^2y^2\dot{y}^2 + aax^2\dot{y}^2 - 4ax^2y\dot{y}^2 + 4y^2x^2\dot{y}^2 = x^2\dot{x}^2$; divide by $aa + xx$, and there arises $aa\dot{y}^2 - 4ay\dot{y}^2 + 4y^2\dot{y}^2 = \frac{x^2\dot{x}^2}{aa + xx}$, extract the Root and $a\dot{y} - 2y\dot{y} = \frac{x\dot{x}}{\sqrt{aa + xx}}$ whence the Fluent is $ay - yy = \pm\sqrt{aa + xx}$.

Ex. 21.

Let $\dot{z}\dot{y} + z\ddot{y} = a\ddot{x}$, then its Fluent is $z\dot{y} = a\dot{x}$.

Ex. 22.

Suppose $a\dot{y}\dot{x}^2 - b\dot{y}\dot{x}^2 = bx\dot{x}\ddot{y} + b\dot{x}^3$, or $a\dot{y}\dot{x}^2 - b\dot{y}\dot{x}^2 - bx\dot{x}\ddot{y} = b\dot{x}^3$, supposing $\dot{x}$ invariable. Then the Fluent will be $ay\dot{x}^2 - bx\dot{x}\dot{y} = bx\dot{x}^2$.

Ex. 23.

Let the Equation be $a\dot{z}\dot{z}^2 + aa\dot{z}\dot{y} - ab\dot{z}\dot{x} = bc\dot{z}\dot{z} - abc\ddot{x}$; then the Fluent is $\frac{az^2\dot{z}}{2} + aay\dot{z} - abx\dot{z} = bcz\dot{z} - abc\dot{x}$; supposing $\dot{z}$ invariable.

II.

If the Quantities cannot be so separated, but one or both of the *Fluxions* contain'd in the Equation will be affected with *both* the *variable Quantities*; then reduce the Equation so that one of the Fluxions alone may possess one Side of the Equation, and the other Fluxion affected with both the variable Quantities be on the other Side, in simple Terms. Then,

1. Range

1. Range all the Quantities that are on the second Side of the Equation so, that all the fluxional Quantities affected only with its *own Fluent*, may stand *horizontally* at the Top, proceeding regularly according to the Indices either increasing or decreasing, according as the Fluent is to be had in an ascending or descending Series; and all the Terms affected with the other flowing Quantity stand *perpendicularly* on the *left* Hand according to their Indices.

2. Begin at the left Hand, and find the Fluent of the *first* Term of the *horizontal* Row, for the first Term of the Fluent; then substitute this instead of the other variable Quantity in all the Terms of the *perpendicular* Row, writing their new Values over against them under their proper Indices in the *horizontal* Row; then proceed to find the second, third, *&c.* Term of the Fluent; by summing up all the fluxionary Quantities of the same Index into one Term, and then finding its Fluent; all which Terms of the Fluent are to be gradually substituted for the Powers of the other Quantity as you go along.

3. And this Operation may be perform'd various Ways, by *assuming* any given Quantity for the first Term, or perhaps for some other Term of the Fluent, and oftentimes it will be *necessary* to do so. If any of the Terms be divided by the *first Power* of its flowing Quantity, it will sometimes be necessary to substitute for this Quantity, the Sum or Difference of a given Quantity and another variable Quantity; and then reduce the Terms to the prescribed Form. And this is the *Newtonian Rule* for finding the *Fluent*.

Ex. 24.

Let $a\dot{y} + x\dot{y} + a\dot{x} + x\dot{x} = y\dot{x} + \frac{ay^3\dot{x} + xy^3\dot{x}}{a^3} - \frac{ax^2\dot{x} - x^3\dot{x}}{a^2}$, to find y. Divide by $a + x$ and reduce

duce the Equation, then $\dot{y} = \dot{x} + \frac{y\dot{x}}{a+x} + \frac{y^3\dot{x}}{a^3} - \frac{x^2\dot{x}}{a^2} = \dot{x} - \frac{x^2\dot{x}}{aa} + \frac{y^3\dot{x}}{a^3} + \frac{y\dot{x}}{a} - \frac{yx\dot{x}}{aa} + \frac{yx^2\dot{x}}{a^3}$ &c. then the Work will be as follows.

	$\dot{x} \quad * \quad -\frac{x^2\dot{x}}{aa}$
$+\frac{y\dot{x}}{a}$	$+\frac{x\dot{x}}{a} + \frac{x^2\dot{x}}{2aa} - \frac{x^3\dot{x}}{2a^3}$ &c.
$-\frac{yx\dot{x}}{aa}$	$-\frac{x^2\dot{x}}{aa} - \frac{x^3\dot{x}}{2a^3}$ &c.
$+\frac{yx^2\dot{x}}{a^3}$ &c.	$+\frac{x^3\dot{x}}{a^3}$ &c.
$+\frac{y^3\dot{x}}{a^3}$	$+\frac{x^3\dot{x}}{a^3}$ &c.

$$\dot{y} = \dot{x} + \frac{x\dot{x}}{a} - \frac{3x^2\dot{x}}{2aa} + \frac{x^3\dot{x}}{a^3} \text{ \&c.}$$

$$y = x + \frac{x^2}{2a} - \frac{x^3}{2aa} + \frac{x^4}{4a^3} \text{ \&c.} = \text{Fluent.}$$

In this Example I write $\dot{x} - \frac{x^2\dot{x}}{aa}$ horizontally and $\frac{y\dot{x}}{a} - \frac{yx\dot{x}}{aa} + \frac{yx^2\dot{x}}{a^3}$ &c. $+ \frac{y^3\dot{x}}{a^3}$ perpendicularly; then I bring down $\dot{x}$ into the Value of $\dot{y}$, and get its Fluent x and put it into the Value of y. Then I substitute x for y in each Term of the perpendicular Row, placing their respective Values $\frac{x\dot{x}}{a}$, $-\frac{x^2\dot{x}}{aa}$, $+\frac{x^3\dot{x}}{a^3}$, and $+\frac{x^3\dot{x}}{a^3}$ against each, under the proper Powers of x.

Then

Then I bring down the Term in the second Place $+\frac{x\dot{x}}{a}$, and write its Fluent $\frac{x^2}{2a}$ in the Value of y; then I write $\frac{x^2}{2a}$ for y in the Terms $+\frac{y\dot{x}}{a}-\frac{yx\dot{x}}{aa}$, &c. and the Refult $\frac{x^2\dot{x}}{2a^2}$, $-\frac{x^3\dot{x}}{2a^3}$ I put over againſt them as before. Then I take the Sum of the Terms in the third Place $-\frac{x^2\dot{x}}{aa}+\frac{x^2\dot{x}}{2aa}-\frac{x^2\dot{x}}{aa}$, and ſet this Sum $-\frac{3x^2\dot{x}}{2aa}$ underneath, and get its Fluent $-\frac{x^3}{2aa}$ which I ſubſtitute for y as before. And thus I proceed as far as I pleaſe.

Ex. 25.

Let $\dot{y}=\dot{x}-3x\dot{x}+y\dot{x}+x^2\dot{x}+yx\dot{x}$, to find y.

Here x the firſt Term being found and written for y, you get $-2x\dot{x}$, and its Fluent $-xx$ for the ſecond Term, which being alſo written for y, you'll get $+x^2\dot{x}$, and its Fluent $\frac{x^3}{3}$ for the third Term; and ſo on as below.

	$+\dot{x}-3x\dot{x}+x^2\dot{x}$
$+y\dot{x}$ $+yx\dot{x}$	$+x\dot{x}-x^2\dot{x}+\frac{1}{3}x^3\dot{x}-\frac{1}{6}x^4\dot{x}$ &c. $+x^2\dot{x}-x^3\dot{x}+\frac{1}{3}x^4\dot{x}$ &c.
	$\dot{y}=\dot{x}-2x\dot{x}+x^2\dot{x}-\frac{2}{3}x^3\dot{x}+\frac{1}{6}x^4\dot{x}$ &c. $y=x-xx+\frac{1}{3}x^3-\frac{1}{6}x^4+\frac{1}{30}x^5$ &c.

Otherwiſe thus.

Here I take a for the firſt Term of y, and writing it inſtead of y, I get $a\dot{x}+\dot{x}$ the next Term, which being

being again written for y, I get the third Term $ax^2 - x^2$, and ſo on; ſee the Work.

	$+\dot{x} - 3x\dot{x} + x^2\dot{x}$
$+y\dot{x}$	$a\dot{x} \begin{smallmatrix}+a\\+1\end{smallmatrix} x\dot{x} \begin{smallmatrix}+a\\-1\end{smallmatrix} x^2\dot{x} \begin{smallmatrix}+\frac{2}{3}a\\+\frac{1}{3}\end{smallmatrix} x_3\dot{x}$ &c.
$+yx\dot{x}$	$+ax\dot{x} \begin{smallmatrix}+a\\+1\end{smallmatrix} x^2\dot{x} \begin{smallmatrix}+a\\-1\end{smallmatrix} x^3\dot{x}$ &c.
$\dot{y} = \begin{smallmatrix}+a\\+1\end{smallmatrix} \dot{x} \begin{smallmatrix}+2a\\\mp 2\end{smallmatrix} x\dot{x} \begin{smallmatrix}+2a\\+1\end{smallmatrix} x^2\dot{x} \begin{smallmatrix}+\frac{5}{3}a\\-\frac{2}{3}\end{smallmatrix} x^3\dot{x}$ &c.	
$y = a \begin{smallmatrix}+a\\+1\end{smallmatrix} x \begin{smallmatrix}+a\\-1\end{smallmatrix} x^2 \begin{smallmatrix}+\frac{2}{3}a\\+\frac{1}{3}\end{smallmatrix} x^3 \begin{smallmatrix}+\frac{5}{12}a\\-\frac{1}{6}\end{smallmatrix} x^4$ &c.	

Ex. 26.

Suppoſe this Equation $\dot{y} = -3x\dot{x} + 3yx\dot{x} + y^2\dot{x} - y^2x\dot{x} + y^3\dot{x} - y^3x\dot{x}$ &c. $+ 6yx^2\dot{x} - 6x^2\dot{x} + 8yx^3\dot{x} - 8x^3\dot{x} + 10yx^4\dot{x} - 10x^4\dot{x}$ &c. To find the Value of y as far as 7 Dimenſions of x.

I place the Terms in Order according to the following Table, and then I work as before; and moreover I ſubjoin the Square and Cube of the Value of y gradually produc'd, to be ſubſtituted by Degrees into their proper Places towards the right Hand, in the Values of the Marginals on the left, as follows.

	$-3x\dot{x}-6x^2\dot{x}-8x^3\dot{x}-10x^4\dot{x}-12x^5\dot{x}-14x^6\dot{x}$ &c.
$+3yx\dot{x}$	$-\frac{9}{2}x^3\dot{x}-6x^4\dot{x}-\frac{75}{8}x^5\dot{x}-\frac{273}{20}x^6\dot{x}$ &c.
$+6yx^2\dot{x}$	$-9x^4\dot{x}-12x^5\dot{x}-\frac{75}{4}x^6\dot{x}$ &c.
$+8yx^3\dot{x}$	$-12x^5\dot{x}-16x^6\dot{x}$ &c.
$+10yx^4\dot{x}$	$-15x^6\dot{x}$ &c.
&c.	
$+yy\dot{x}$	$+\frac{9}{4}x^4\dot{x}+6x^5\dot{x}+\frac{107}{8}x^6\dot{x}$ &c.
$-y^2x\dot{x}$	$-\frac{9}{4}x^5\dot{x}-6x^6\dot{x}$ &c.
&c.	
$+y^3\dot{x}$	$-\frac{27}{8}x^6\dot{x}$ &c.

$\dot{y}=-3x\dot{x}-6x^2\dot{x}-\frac{25}{2}x^3\dot{x}-\frac{91}{4}x^4\dot{x}-\frac{333}{8}x^5\dot{x}-\frac{367}{5}x^6\dot{x}$ &c.

$y=-\frac{3}{2}x^2-2x^3-\frac{25}{8}x^4-\frac{91}{20}x^5-\frac{111}{16}x^6-\frac{367}{35}x^7$ &c.

$y^2=\frac{9}{4}x^4+6x^5+\frac{107}{8}x^6$ &c.

$y^3=-\frac{27}{8}x^6$ &c.

Ex. 27.

Let $\dot{y}=\frac{y\dot{x}}{xx}+\frac{\dot{x}}{xx}+3\dot{x}+2x\dot{x}-\frac{4\dot{x}}{x}$; to find y in a descending Series. Here the Terms in the horizontal Row must be placed to proceed from the greater Indices to the lesser: And the Work will be as below.

	$+2x\dot{x}+3\dot{x}-\frac{4\dot{x}}{x}+\frac{\dot{x}}{xx}$
$+\frac{y\dot{x}}{xx}$	$\dot{x}+\frac{4\dot{x}}{x}\quad * -\frac{\dot{x}}{x^3}+\frac{\dot{x}}{2x^4}$ &c.

$\dot{y}=2x\dot{x}+4\dot{x}+0+\frac{\dot{x}}{xx}-\frac{\dot{x}}{x^3}+\frac{\dot{x}}{2x^4}$ &c.

$y=x^2+4x.\quad * -\frac{1}{x}+\frac{1}{2x^2}-\frac{1}{6x^3}$ &c.

Here obſerve, that any given Quantity might have been inſerted between the Terms $4x$ and $-\frac{1}{x}$; and ſo y might be extracted an infinite Variety of ways. But if that fluxionary Term had not vaniſh'd, then we had been obliged to ſubſtitute $a+z$ or $a-z$ for x in the given Equation before it could be reſolved.

Ex. 28.

Suppoſe the given Equation $x^2\dot{y} = x\dot{x} - \dot{x} - yx^2\dot{x}$; by Reduction $\dot{y} = \frac{\dot{x}}{x} - y\dot{x} - \frac{\dot{x}}{xx}$, then the Work will be very eaſily performed as in the following Table.

	$-\frac{\dot{x}}{xx} + \frac{\dot{x}}{x}$
$-y\dot{x}$	$-\frac{\dot{x}}{x}$
$\dot{y} = -\frac{\dot{x}}{xx} + 0$	
$y = \frac{1}{x}$	

III.

When the given Equation contains firſt Fluxions alone, or if it contains *firſt*, *ſecond*, or *third*, &c. *Fluxions*, as in the following Example, where $2ax\dot{y}^3 - ay^2\ddot{x} + 2x^2\dot{y}^2 - 2y^2\dot{x}^2 = 0$ to find x expreſſed by y and given Quantities; it will be reſolved by the following general Method.

1. Make the Equation $= 0$; and *aſſume* an *indetermin'd Series* to repreſent the Series required; as $x = Ay^n + By^{n+r} + Cy^{n+s} + Dy^{n+t}$, &c. wherein the

the Indices n, $n+r$, $n+s$ continually increase if y be very small, or decrease if it be great.

2. To find the first Index n; substitute into the given Equation the *first Term* Ay^n, its *Fluxion*, and *second Fluxion*, &c. instead of x, $\dot{x}$, and $\ddot{x}$, &c. (if they be there) and then you'll have a new Equation, as $2aAy^n - \overline{nn-n} \times aAy^n + 2A^2y^{2n} - 2n^2A^2y^{2n} = 0$, supposing $\dot{y}=1$. Make two (or more) of the Terms *equal* to nothing that have the *least Indices* equal to one another, for an *ascending Series*; or those that have the *greatest Indices* equal, for a *descending Series:* then by equating their Indices or else their Coefficients, n will be found. Or if there happen to be only *one Term* with such least or greatest Index, make its *Coefficient* $=0$, which will destroy that Term, and perhaps give the Value of n: Thus, in this Example for an ascending Series, you'll have $2aAy^n - \overline{nn-n} \times aAy^n = 0$, or $nn-n=2$, whence $n=2$.

3. For the other Indices; *substitute* the Value of n into the foregoing Equation, and you'll have $2aAy^2 - 2aAy^2 + 2A^2y^4 - 8A^2y^4 = 0$. Then take the *least Index* in an *ascending Series*, or the *greatest* in a *descending* one from each of the rest; and find all the possible Numbers that result by *adding* all these Remainders to themselves and to one another as oft as possible, and then you have r, s, t, &c. here $4-2=2$ only one Remainder; then 2, 4, 6, *&c.* $=r$, s, t, &c. And therefore the Series is $Ay^2 + By^4 + Cy^6 + Dy^8$ *&c.* $=x$.

4. For determining the Coefficients A, B, C, *&c.* *substitute* into the given Equation the *Values* of x, $\dot{x}$, $\ddot{x}$, &c. expressed by the foregoing Series $Ay^2 + By^4 + Cy^6$, *&c.* and put the *Sum* of the *Coefficients* of every several Power of y *equal* to nothing, and thence A, B, C, *&c.* will be gradually found. Thus $\dot{x} = 2Ay + 4By^3 + 6Cy^5$, *&c.* $\ddot{x} = 2A + 12By^2 + 30Cy^4$, *&c.* where $\dot{y}=1$, these being substituted, the Work will be perform'd as below.

5. If

5. If the firſt Equation for the Coefficients be an *adfected* Equation containing ſeveral *different Powers* of A, then that Equation will afford ſeveral Roots or Values of A; and as many different Roots ſo many different Series may be obtain'd. And if A has ſeveral equal Values or Roots in this Equation, then you muſt divide the leaſt Remainder above by that Number (of equal Roots of A, one of which you aſſume for its Value;) then proceed as in Art. 3. taking this Quotient for another Remainder.

6. If a Series be required to be expreſs'd in *Terms* of that *Quantity* whoſe 2d, 3d Fluxion, *&c.* is in the Equation; it muſt firſt be got in Terms of the *other* Quantity that has no ſecond, third, *&c.* Fluxion; and then the Series reverted.

SCHOL. As there are fluxionary Equations that admit of ſeveral Solutions, and may have the Root expreſs'd various ways; ſo there are Equations that cannot be reſolved at all as being impoſſible; and others that are very difficult to be reſolved, and may require the utmoſt Skill of the Analiſt, and ſometimes a different Proceſs from theſe Rules.

Ex. 29.

Let $2ax\dot{y}^2 - ay^2\ddot{x} + 2x^2\dot{y}^2 - 2y^2\dot{x}^2 = 0$; to find x expreſſed by y and given Quantities.

The Form of the Series found by the foregoing Rule is $x = Ay^2 + By^4 + Cy^6 + Dy^8$ *&c.* This and it's Fluxions being ſubſtituted into the given Equation will be as follows.

$$
\begin{array}{l|lllll}
+2ax\dot{y}^2 & +2aAy^2 & +\ 2aBy^4 & +\ 2aCy^6 & +\ 2aDy^8 & \&c. \\
-ay^2\ddot{x} & -2aAy^2 & -24aBy^4 & -60aCy^6 & -112aDy^8 & \&c. \\
+2x^2\dot{y}^2 & & +\ 2A^2y^4 & +4ABy^6 & \left.\begin{array}{l}+2BB\\+4AC\end{array}\right\} y^8 & \&c. \\
-2y^2\dot{x}^2 & & -\ 8A^2y^4 & -32ABy^6 & \left.\begin{array}{l}-32BB\\-48AC\end{array}\right\} y^8 & \&c.
\end{array}
$$

Then

Then equating their respective Coefficients; $2aA - 2aA = 0$, therefore A may be taken at Pleasure. Again $-22aB - 6A^2 = 0$, thence $B = \frac{-3A^2}{11a}$. After the same Manner $C = \frac{42A^3}{319aa}$, and $D = \frac{-10164A^4}{17545a^3}$ &c. Whence $x = Ay^2 - \frac{3A^2}{11a}y^4 + \frac{42A^3}{319aa}y^6 - \frac{10164A^4}{17545a^3}y^8$ &c.

Otherwise thus for a descending Series.

Make the Terms $2A^2y^{2n} - 2n^2A^2y^{2n} = 0$, or $1 - nn = 0$, whence $n = 1$; this substituted for n will produce $2aAy + 2A^2y^2 - 2A^2y^2 = 0$. Take the greatest Index 2 from the rest, which is 1, and there remains -1; therefore $r, s, t,$ &c. are $-1, -2, -3,$ &c. and the Series is $Ay + B + Cy^{-1} + Dy^{-2}$ &c. and the Operation will be as follows.

$$\begin{array}{l|lllll}
+2ax\dot{y}^2 & & +2aAy & +2aB & +2aCy^{-1} & \&c.\\
-ay^2\ddot{x} & & & & -2aCy^{-1} & \&c.\\
+2x^2\dot{y}^2 & +2A^2y^2 & +2ABy & \begin{array}{l}+2BB\\+4AC\end{array} & \begin{array}{l}+4BC\\+4AD\end{array}y^{-1} & \&c.\\
-2y^2\dot{x}^2 & -2A^2y^2 & & +4AC & +8ADy^{-1} & \&c.
\end{array}$$

$$\text{or}\left\{\begin{array}{l}+2A^2\\-2A^2\end{array}\right.y^2 \begin{array}{l}+2aA\\+4AB\end{array}y \begin{array}{l}+2aB\\+2BB\\+8AC\end{array} \begin{array}{l}+2BC\\+12AD\end{array}y^{-1}\ \&c. = 0.$$

Hence, equaling the Coefficients; $2A^2 - 2A^2 = 0$, and A may be again taken at Pleasure; likewise $B = \frac{-a}{2}$, $C = \frac{aa}{4A}$, $D = \frac{a^3}{24A^2}$, &c. whence the Series is known, and $x = Ay - \frac{a}{2} + \frac{aa}{4Ay} + \frac{a^3}{24A^2y^2}$ &c.

Ex. 30.

Let $a^3\ddot{y} - a^2y\dddot{y} + yy\ddot{x}^2 = 0$, to find y in an ascending Series.

Assume

Assume $y = Ax^n + Bx^{n+r} + Cx^{n+s}$ &c. and substituting the first Term and its Fluxions for y and its Fluxions, we have $n.\overline{n-1}.\overline{n-2}.a^2Ax^{n-3} - n^2.\overline{n-1} \times a^2A^2x^{2n-3} + nA^2x^{2n-1} = 0$. Since the Index $\overline{n-3}$ is the least Index, make its Coefficient $n.\overline{n-1}.\overline{n-2} = 0$, and take one of the Roots $n = 2$. Whence the Indices will be -1, 1, 3, subtract the least, -1 from the rest, and there remains 2, 4; therefore r, s, $t = 2, 4, 6$ &c. whence $y = Ax^2 + Bx^4 + Cx^6$ &c. and the rest of the Work is as follows.

$$\begin{array}{l|llll}
+a^3\dddot{y} & +24a^3Bx & +120a^3Cx^3 & +336a^3Dx^5 & \text{\&c.} \\
-a^2\dot{y}\ddot{y} & -4a^2A^2x & -32a^2ABx^3 & -72a^2ACx^5 & \text{\&c.} \\
 & & & -48a^2B^2 & \\
+y\dot{y}x^2 & & +2A^2x^3 & +6ABx^5 & \text{\&c.}
\end{array}$$

Hence $24a^3B = 4a^2A^2$, and A may be taken at pleasure; let $A = \frac{1}{2a}$ then $B = \frac{1}{24a^3}$, also $C = \frac{1}{720a^5}$, $D = \frac{-1}{4480a^7}$ &c. Whence $y = \frac{x^2}{2a} + \frac{x^4}{24a^3} + \frac{x^6}{720a^5} - \frac{x^8}{4480a^7}$ &c.

Otherwise for a descending Series.

Let $2n-1$ and $n-3$ be supposed to be the greatest Indices, then $n-3 = 2n-1$, and $n = -2$, and these Indices will be -5, -7; and taking -5 from -7 the Remainder is -2; therefore r, s, $t = -2, -4, -6$ &c. Wherefore $y = Ax^{-2} + Bx^{-4} + Cx^{-6}$ &c. the rest of the Work will be thus,

$$\begin{array}{l|llll}
+a^3\dddot{y} & -24a^3Ax^{-5} & -120a^3Bx^{-7} & -336a^3Cx^{-9} & \text{\&c.} \\
-a^2\dot{y}\ddot{y} & & +12a^2A^2x^{-7} & +64a^2ABx^{-9} & \text{\&c.} \\
+y\dot{y}x^2 & -2A^2x^{-5} & -6ABx^{-7} & \left.\begin{array}{l}-8AC \\ -4BB\end{array}\right\} x^{-9} & \text{\&c.}
\end{array}$$

Therefore $A = -12a^3$, $B = 36a^5$, $C = -\frac{1116}{10}a^7$ &c.
And $y = -12a^3x^{-2} + 36a^5x^{-4} - \frac{1116}{10}a^7x^{-6}$ &c.

Ex. 31.

Ex. 31.

Let the Equation be $\dot{x} = \frac{1}{2}y\dot{y} - 4y^2\dot{y} + 2x^{\frac{1}{2}}y\dot{y} - \frac{4}{5}x^2\dot{y} + 7y^{\frac{5}{2}}\dot{y} + 2y^3\dot{y}$, to find x in y ascending.

Let $x = Ay^n + By^{n+r} + Cy^{n+s}$ &c; and substituting Ay^n and its Fluxion for x and $\dot{x}$, there arises $-nAy^{n-1} + \frac{1}{2}y - 4y^2 + A^{\frac{1}{2}}y^{\frac{n}{2}+1} - \frac{4}{5}A^2y^{2n} + 7y^{\frac{5}{2}} + 2y^3 = 0$. Suppose $n-1$ and 1 to be the least Indices, and you will have $-nAy^{n-1} + \frac{1}{2}y = 0$, and $n-1 = 1$, or $n = 2$, and all the Indices will be $1, 2, 4, 2\frac{1}{2}, 3$. Take 1 from the Rest and the Remainders are, $1, 1\frac{1}{2}, 2, 3$; whence r, s, t, &c. are $1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3$, &c. and $x = Ay^2 + By^3 + Cy^{3\frac{1}{2}} + Dy^4$ &c. hence,

$-\dot{x}$	$-2Ay$	$-3By^2$	$-3\frac{1}{2}Cy^{2\frac{1}{2}}$	$-4Dy^3$ &c.
$+\frac{1}{2}y$	$+\frac{1}{2}y$			
$-4y^2$		$-4y^2$		
$+2yx^{\frac{1}{2}}$		$+2A^{\frac{1}{2}}y^2$	$*$	$+\frac{By^3}{A^{\frac{1}{2}}}$
$-\frac{4}{5}x^2$		$*$	$*$	$*$
$+7y^{\frac{5}{2}}$			$+7y^{\frac{5}{2}}$	$*$
$+2y^3$				$+2y^3$

therefore $A = \frac{1}{4}$, likewise $B = -1$, $C = 2$, $D = 0$, &c. and the Series is $x = \frac{1}{4}y^2 - y^3 + 2y^{\frac{7}{2}}$ &c.

Ex. 32.

Let $\dot{z}^3 - c\dot{z}^2 - 2x^2\dot{z} - c^2\dot{z} + 2x^3 + c^3 = 0$, to find z in a Series of x ascending. Put $z = Ax^n + Bx^{n+r} + Cx^{n+s}$ &c. and by Substitution according to the Rule, we have $n^3A^3x^{3n-3} - cn^2A^2x^{2n-2} - 2nAx^{n+1} - nccAx^{n-1} + 2x^3 + c^3 = 0$; and supposing the least Indices $n-1$ and 0 to be equal, we have $n = 1$. And all these Indices will become $0, 2, 3$: and substracting the least 0, the Remainders will be $2, 3$; whench r, s, t, &c. $= 2, 3, 4$, &c. and the Series is $z = Ax + Bx^3 + Cx^4 + Dx^5$ &c. and the Operation as follows,

$$\begin{array}{l|lllll}
+\dot{z}^3 & +A^3 & +9A^2Bx^2 & +12A^2Cx^3 & \begin{matrix}+15A^2D\\+27AB^2\end{matrix}\,x^4 & \text{\&c.}\\
-c\dot{z}^2 & -cA^2 & -6cABx^2 & -8cACx^3 & \begin{matrix}-9cB^2\\-10cAD\end{matrix}\,x^4 & \text{\&c.}\\
-2x^2\dot{z} & & -2A^2x^2 & * & -6B\,x^4 & \text{\&c.}\\
-c^2\dot{z} & -c^2A & -3c^2Bx^2 & -4c^2Cx^3 & -5c^2D\,x^4 & \text{\&c.}\\
+2x^3 & & & +2x^3 & * & \\
+c^3 & +c^3 & & & &
\end{array}$$

Here $A^3 - cA^2 - c^2A + c^3 = 0$, and $A = +c$ or $-c$: If $A = +c$ then B will be infinite, therefore $A = -c$, and then $B = -\frac{1}{6c}$, $C = -\frac{1}{8cc}$, $D = 0$, &c. And $z = -cx - \frac{x^3}{6c} - \frac{x^4}{8cc}$ &c.

If you take $A = +c$; then since the Equation $A^3 - cA^2 - c^2A + c^3 = 0$, contains two Roots $= c$; therefore divide 2 (the least Remainder) by 2 (the Number of equal Roots) and the Quotient is 1. Then by Help of the Remainders 1, 2, 3, &c. you will get the Series $z = Ax + Bx^2 + Cx^3$, &c. with this proceed as with the former, and you will get other Series for the Value of z, wherein B may be taken at Pleasure.

Ex. 33.

Suppose $e\dot{y} + fz^m\dot{y} + dyz^{m-1}\dot{z} = z^p\dot{z}$, to find y, Assume $y = Az^n + Bz^{n+r} + Cz^{n+s}$ &c. then puting Az^n and its Fluxion for y and $\dot{y}$; and you have $enAz^{n-1} + fnAz^{m+n-1} + dAz^{m+n-1} - z^p = 0$; take the Indices $n-1=p$, then $n=p+1$, and the Indices become p, $p+m$, and the common Difference $= m$, and $r, s, t = m, 2m, 3m$, &c. Whence $y=Az^{p+1} + Bz^{p+m+1} + Cz^{p+2m+1}$ &c. therefore,

$$\begin{array}{r|lllll}
+e\dot{y} & \overline{p+1}.eAz^p & +\overline{p+1+m}.eBz^{p+m} & +\overline{p+1+2m}.eCz^{p+2m} & \&c. \\
+fz^m\dot{y} & & +\overline{p+1}.fAz^{p+m} & +\overline{p+1+m}.fBz^{p+2m} & \&c. \\
+dyz^{m-1}\dot{z} & & +\quad dAz^{p+m} & +\quad dBz^{p+2m} & \&c. \\
-z^p\dot{z} & -\ z^p & & &
\end{array}$$

Hence $A=\frac{1}{\overline{p+1}.e}$, $B=-\frac{d+\overline{p+1}.f}{\overline{p+1+m}.e}A$, $C=-\frac{d+\overline{p+1+m}.f}{\overline{p+1+2m}.e}B$, &c. and putting $\varepsilon=p+1$, and $\delta=\frac{d}{f}+\overline{p+1}$, then $y=\frac{z^\varepsilon}{\varepsilon e}-\frac{\delta f}{\overline{\varepsilon+m}.e}Az^{\varepsilon+m}-\frac{\overline{\delta+m}.f}{\overline{\varepsilon+2m}.e}Bz^{\varepsilon+2m}-\frac{\overline{\delta+2m}.f}{\overline{\varepsilon+3m}.e}Cz^{\varepsilon+3m}$ &c.

Ex. 34.

Let $\dot{x}=\overline{\alpha+\beta z^m}^{\mu}z^\pi\dot{z}$, to find x by a Series of z. Suppose $x=y\times\overline{\alpha+\beta z^m}^{\mu}$, this in Fluxions gives $\dot{x}=\dot{y}\times\overline{\alpha+\beta z^m}^{\mu}+\mu m\beta yz^{m-1}\dot{z}\times\overline{\alpha+\beta z^m}^{\mu-1}=\overline{\alpha+\beta z^m}^{\mu}\times z^\pi\dot{z}$, that is $\alpha\dot{y}+\beta z^m\dot{y}+\mu m\beta yz^{m-1}\dot{z}-\alpha z^\pi\dot{z}-\beta z^{\pi+m}\dot{z}=0$. Let $y=Az^n+Bz^{n+r}+Cz^{n+s}$ &c. then by Substitution $\alpha nAz^{n-1}+n\beta Az^{n+m-1}+\mu m\beta Az^{n+m-1}-az^\pi-\beta z^{\pi+m}=0$. And putting the Indices $n-1=\pi$, then $n=\pi+1$, and these Indices are π, $\pi+m$, and the common Difference m, whence $y=Az^{\pi+1}+Bz^{\pi+1+m}+Cz^{\pi+1+2m}$ &c.

And

$$\begin{array}{r|lllll}
\alpha\dot{y} & \overline{\pi+1}.\alpha Az^\pi & +\overline{\pi+1+m}.\alpha Bz^{\pi+m} & +\overline{\pi+1+2m}.\alpha Cz^{\pi+2m} & \&c. \\
+\beta z^m\dot{y} & & +\overline{\pi+1}.\beta A & +\overline{\pi+1+m}.\beta B & \&c. \\
+\mu m\beta yz^{m-1}\dot{y} & & +\mu m\beta A & +\mu m\beta B & \&c. \\
-\alpha z^\pi\dot{z}-\beta z^{\pi+m}\dot{z} & -\ \alpha & -\ \beta & &
\end{array}$$

Hence $A = \frac{1}{\pi + 1}$, $B = -\frac{\mu m \beta}{\pi + 1 . \pi + 1 + m . \alpha}$, $C = \frac{\overline{\mu m + \pi + 1 + m} \times \beta^2 \mu m}{\pi + 1 . \pi + 1 + m . \pi + 1 + 2m . \alpha^2}$ &c. whence putting $q = \frac{\beta z^m}{\alpha}$, $\varepsilon = \pi + 1$, $\delta = \varepsilon + \mu m$. P, Q, R, each preceding Term with its Sign, then $x = \overline{\alpha + \beta z^m}^{\mu} \times : \frac{z^{\varepsilon}}{\varepsilon} - \frac{\mu m q}{\varepsilon + m} P - \frac{\overline{\delta + m} . q}{\varepsilon + 2m} Q - \frac{\overline{\delta + 2m} . q}{\varepsilon + 3m} R -$ &c: $=$ Fl: $\overline{\alpha + \beta z^m}^{\mu} z^{\pi} \dot{z}$.

Ex. 35.

Let $\dot{x} = \overline{e + fz^n + gz^{2n} + hz^{3n} \text{\&c.}} \times \overline{\alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n} \text{\&c.}}^{\mu} \times z^{\pi} \dot{z}$, to find x by z. Suppose $x = y \times \overline{\alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n} \text{\&c.}}^{\mu + 1}$ this put into Fluxions, and then the whole divided by $\overline{\alpha + \beta z^n + \gamma z^{2n}}^{\mu}$ &c. you have $z^{\pi} \dot{z} \times \overline{e + fz^n + gz^{2n} \text{\&c}}$ $= \dot{y} \times \overline{\alpha + \beta z^n + \gamma z^{2n} \text{\&c}} + \overline{\mu + 1} . y \times \overline{n \beta z^{n-1} \dot{z} + 2n \gamma z^{2n-1}}$ &c. Let $y = Az^m + Bz^{m+r} + Cz^{m+s}$ &c. then Az^m and its Fluxion being substituted for y and $\dot{y}$, for the first Term of each compound Quantity in the foregoing Equation, and there will be $-ez^{\pi} + m\alpha Az^{m-1} + \overline{\mu + 1} . n\beta Az^{m+n-1}$ &c. $= 0$; and making the Indices π and $m - 1$ equal, then $m = \pi + 1$, and the other Index is $\pi + 1 + n$, and the common Difference n, whence $y = Az^{\pi+1} + Bz^{\pi+1+n} + Cz^{\pi+1+2n}$ &c. And putting $\pi + 1 = p$, then $\dot{y} = pAz^{\pi} + \overline{p + n} . Bz^{\pi+n} + \overline{p + 2n} \times Cz^{\pi+2n}$ &c. put $m = \mu + 1$, then

$$\begin{array}{r|llll}
-ez^{\pi}-fz^{\pi+n}\ \&c. & -ez^{\pi} & -fz^{\pi+n} & -gz^{\pi+2n}\ \&c. & \\
+\alpha\dot{y} & +p\alpha A & +\overline{p+n}.\alpha B & +\overline{p+2n}.\alpha C & \&c. \\
+\beta z^{n}\dot{y} & & +p\beta A & +\overline{p+n}.\beta B & \&c. \\
+\gamma z^{2n}\dot{y} & & & +\ p\gamma A & \&c. \\
\&c. & & & & \\
+mn\beta y z^{n-1} & & +mn\beta A & +\ mn\beta B & \&c. \\
+2mn\gamma y z^{2n-1} & & & +\ 2mn\gamma A & \&c. \\
\&c. & & & &
\end{array}$$

Hence $A=\frac{e}{p\alpha}$, $B=\frac{f\left.\begin{matrix}-p\\-mn\end{matrix}\right\}\beta A}{\overline{p+n}.\alpha}$, $C=\frac{g\left.\begin{matrix}-p\\-2mn\end{matrix}\right\}\gamma A\left.\begin{matrix}-\overline{p+n}\\-mn\end{matrix}\right\}\beta B}{\overline{p+2n}.\alpha}$

$D=\frac{h\left.\begin{matrix}-p\\-3mn\end{matrix}\right\}\delta A\left.\begin{matrix}-\overline{p+n}\\-2mn\end{matrix}\right\}\gamma B\left.\begin{matrix}-\overline{p+2n}\\-mn\end{matrix}\right\}\beta C}{\overline{p+3n}.\alpha}$, $E=\frac{k\left.\begin{matrix}-p\\-4mn\end{matrix}\right\}\epsilon A\left.\begin{matrix}-\overline{p+n}\\-3mn\end{matrix}\right\}\delta B\left.\begin{matrix}-\overline{p+2n}\\-2mn\end{matrix}\right\}\gamma C\left.\begin{matrix}-\overline{p+3n}\\-mn\end{matrix}\right\}\beta D}{\overline{p+4n}.\alpha}$, &c.

And $x=\overline{Az^{p}+Bz^{p+n}+Cz^{p+2n}+Dz^{p+3n}\ \&c.}\times\overline{\alpha+\beta z^{n}+\gamma z^{2n}+\delta z^{3n}\ \&c.}^{\mu+1}$

IV.

The *Fluent* of an *irrational Fluxion* may ſometimes alſo be found by *aſſuming* an *indetermin'd Series* as in the laſt Rule.

Ex. 36.

Suppoſe $\dot{z}=v^{2}x^{2}\dot{x}$, where $\dot{v}=\frac{a\dot{x}}{y}$, and $y=\sqrt{2ax-xx}$. I take $Av^{2}x^{3}$ for the firſt Term, and aſſume as many Terms of the inferiour Powers of v and x, or their Products, as I think will be ſufficient; for which no general Rule can be given. But you need take no more than the firſt Power of y, becauſe all the Powers above will be expreſs'd by the Powers of x, which are ſuppoſed to be already in the Equation.

tion. Thus I assume $z = Av^2x^3 + Bv^2 + \overline{Cx^2 + Dx + E} \times vy + Fx^3 + Gx^2 + Hx = \text{Flu}^t.\ v^2x^2\dot{x}$. Put this Equation into Fluxions making $\dot{x} = 1$, and writing every where $\frac{a}{y}$ for $\dot{v}$; then reduce it from Fractions, writing $2ax - xx$ for yy where it occurs. Then collect severally all the homologous Terms (or those of the same Powers of all or any of the Quantities x, v, y), thus

$$\begin{array}{lllll} +3Av^2x^2y & +2aAx^3v & +5aCx^2v & +3aDxv & +2aBv \\ -1 & -3C & -2D & -E & +aE \end{array}$$

$$\left.\begin{array}{lll} +3Fx^2y & +aDxy & +aEy \\ +aC & +2G & +H \end{array}\right\} = 0.$$

The respective Coefficients then being equated, there will be found $A = \frac{1}{3}$, $C = \frac{2}{9}a$, $D = \frac{5}{9}aa$, $E = \frac{5}{3}a^3$, $B = -\frac{5}{6}a^3$, $F = -\frac{2}{27}aa$, $G = -\frac{5}{18}a^3$, $H = -\frac{5}{3}a^4$. And thence $z = \frac{x^3v^2}{3} - \frac{5}{6}a^3v^2 + \overline{\frac{2}{9}ax^2 + \frac{5}{9}aax + \frac{5}{3}a^3} \times vy - \frac{2}{27}a^2x^3 - \frac{5}{18}a^3x^2 - \frac{5}{3}a^4x$.

Note, If any of the Quantities B, C, D &c. come out equal to nothing, still the Series will be true, *provided* they don't destroy the Quantity A. But if A *vanish* by reason of some of the other Quantities being nothing; or if they involve some *impossible Equations*, then the Series is *not* true; and you must try again by assuming more Terms of the Powers or Products of x, v, y. But in many Cases it cannot be done in finite Terms.

V.

In a *fluxionaay Equation* where the variable Quantity is very *great*, and you would express the Fluent by an *ascending Series*: Or in any very much *compounded* fluxionary Quantity whose Fluent is required; take a given Quantity extremely *near* equal to the variable Quantity; then instead of that variable Quantity *sub-*

stitute

stitute into the Equation the *Sum* of this given Quantity and a new variable Quantity, and likewise the Fluxion for the Fluxion; then find the *Fluent* in simple Terms, and this will be a *Part* of the whole *Fluent* required; and the Operation *repeated* as often as necessary will give the whole Fluent.

Ex. 37.

Let $\dot{z} = \dot{x}\sqrt{aa + xx}$; suppose r very near equal to x, and put $r + v = x$, and $aa + rr = ss$, then $\dot{x} = \dot{v}$, and $\dot{z} = \dot{v} \times \sqrt{aa + rr + 2rv + vv} = \dot{v}\sqrt{ss + 2rv + vv}$ $= \dot{v} \times : s + \frac{2rv + vv}{2s} - \frac{\overline{2rv + vv}^2}{8s^3} + \frac{\overline{2rv + vv}^3}{16s^5} -$ &c: $= \dot{v} \times : s + \frac{rv}{s} + \frac{ss - rr}{2s^3}v^2 - \frac{ss - rr}{2s^5}rv^3 +$ &c: $= s\dot{v} + \frac{rv\dot{v}}{s} + \frac{aav^2\dot{v}}{2s^3} - \frac{aarv^3\dot{v}}{2s^5}$ &c. Whence $z = sv + \frac{rv^2}{2s} + \frac{aav^3}{6s^3} - \frac{aarv^4}{8s^5}$ &c. Now in this, substitute $-v$ for v, and substract the Result from the last Equation, and then $z = 2sv + \frac{a^2v^3}{3s^3}$ &c. And this is the Part of the Fluent corresponding to the Difference of the Quantities $r + v$ and $r - v$, or to $2v$, that is, to these two different Values of x. Hence if there be assumed successively the Numbers or Quantities b, c, d, e, f, &c. for r, and v be taken extremely small, and it be always $b - v = c + v$, and $c - v = d + v$, and $d - v = e + v$ &c. till f (the least Value of r) be o; then the Sum of all the Parts corresponding to each (collected by the foregoing Series) will be the whole Fluent z required.

Ex. 38.

Ex. 38.

Suppose $\dot{z} = \frac{1}{2}\dot{x}\sqrt{\frac{pp + \overline{d - \frac{bx}{a}}|^2}{2ax - xx}}$; let $r + v = x$.

then $\dot{z} = \frac{1}{2}\dot{v}\sqrt{\frac{pp + dd - \frac{2dbr}{a} - \frac{2dbv}{a} + \frac{bb}{aa} \times \overline{r+v}^2}{2ar + 2av - rr - 2rv - vv}} =$

(putting $ss = pp + dd - \frac{2dbr}{a} + \frac{bbrr}{aa}$, $t = \frac{2bbr}{aa} - \frac{2db}{a}$,

$qq = 2ar - rr$, and $n = 2a - 2r$,) $\frac{1}{2}\dot{v}\sqrt{\frac{ss + tv + \frac{bb}{aa}vv}{qq + nv - vv}}$.

Then dividing the Numerator by the Denominator, and puting $e = \frac{tqq - nss}{q^4}$, $f = \frac{bb}{aaqq} + \frac{ss}{q^4} + \frac{n^2s^2 - ntq^2}{q^6}$

we have $\dot{z} = \frac{1}{2}\dot{v}\sqrt{\frac{ss}{qq} + ev + fv^2 \text{ \&c.}} = \frac{1}{2}\dot{v} \times : \frac{s}{q} + \frac{qe}{2s}v + \frac{4qfs^2 - q^3e^2}{8s^3}v^2$ &c: Whence $z = \frac{sv}{2q} + \frac{qe}{8s}v^2 + \frac{4fs^2 - q^2e^2}{24s^3}qv^3$ &c. In which substituting $-v$ for v, and substracting the Result from the Equation, we have $z = \frac{sv}{q} + \frac{4fs^2 - q^2e^2}{12s^3}qv^3 +$ &c. for that Part of the Fluent belonging to $2\dot{v}$, or to the Difference of the Quantities $r + v$ and $r - v$.

Where it is *difficult* to get *many Terms* of the Series as in this last Problem, v must be taken so much the *smaller*, and the Operations *oftner* repeated before we can obtain the whole Fluent: And the working with Numbers instead of known Letters, may sometimes be preferable, when the Quantities are very complex.

Besides

Besides the *general Rules* before delivered, there are some *particular Rules* which in some Cases will find the Fluent in *finite* Terms. As

VI.

When one of the *variable Quantities* is *wanting* in the Equation: Then *assume* for its Fluxion the Product of the *other* Fluxion and a *new* variable Quantity, which substitute for the other; and you will get an Equation which put into Fluxions, will give the *Value* of the exterminated Fluxion; and then the Fluent will give the *assumed* Quantity; and from thence the *other* Quantity will be had.

Ex. 39.

Let $y\dot{y}^3\dot{x} = a\dot{x}^4 + 2a\dot{x}^2\dot{y}^2 + a\dot{y}^4$, where x is wanting. Assume $\frac{z\dot{y}}{a} = \dot{x}$, and expunging $\dot{x}$, $aazy = z^4 + 2a^2z^2 + a^4$, whence $y = \frac{z^3}{aa} + 2z + \frac{aa}{z}$. In Fluxions $\dot{y} = \frac{3z^2\dot{z}}{aa} + z\dot{z} - \frac{aa\dot{z}}{zz}$, therefore $\frac{z\dot{y}}{a}$ or $\dot{x} = \frac{3z^3\dot{z}}{a^3} + \frac{z^2\dot{z}}{a} - \frac{a\dot{z}}{z}$. Whence the Fluent is $x = \frac{3z^4}{4a^3} + \frac{z^3}{3a} - a \times 2.302585$ Log. z. therefore y being known, z will be known (by the Equation $y = \frac{z^3}{aa} + 2z + \frac{aa}{z}$), and consequently x by the last Equation.

VII.

Sometimes the *Fluent* may be had, by firſt putting the Equation into *Fluxions*, making ſome of the Fluxions *invariable.*

Ex. 40.

Let $\frac{a\dot{x}+y\dot{x}}{\dot{y}}=x+y-\frac{x\dot{y}}{\dot{x}}$. make $\dot{y}$ conſtant, and put the Equation into Fluxions, then $\frac{\overline{a+y}\,\ddot{x}}{\dot{y}}+\dot{x}=\dot{x}+\dot{y}+\frac{x\dot{y}\ddot{x}-\dot{x}^2\dot{y}}{\dot{x}^2}$, and $\frac{\overline{a+y}.\ddot{x}}{\dot{y}}=\frac{x\dot{y}\ddot{x}}{\dot{x}^2}$, whence $\overline{a+y}.\dot{x}^2=x\dot{y}^2$, and $\frac{\dot{x}}{\sqrt{x}}=\frac{\dot{y}}{\sqrt{a+y}}$, and the Fluent is $\sqrt{x}=\sqrt{a+y}$, or $x=a+y$.

VIII.

Sometimes the *Fluent* may be found by *aſſuming* other *variable* Quantities to make up the Fluent, and finding their Values by help of the *given* Equations and their Fluxions.

Ex. 41.

Suppoſe $y\dot{x}-x\dot{x}=a\dot{y}$, To find x.

Divide by $y-x$, and $\dot{x}=\frac{a\dot{y}}{y-x}$. Aſſume v, and ſuppoſe the Fluent $x=a\times 2.302585$ Log. $\overline{y-x+v}$. then will $\dot{x}=\frac{a\dot{y}-a\dot{x}+a\dot{v}}{y-x+v}$ (ſee Prob. 2. Sect. II.), and by multiplying, $y\dot{x}-x\dot{x}+v\dot{x}=a\dot{y}-a\dot{x}+a\dot{v}$, from

from this substract the given Equation $y\dot{x} - x\dot{x} = a\dot{y}$, and there remains $v\dot{x} = -a\dot{x} + a\dot{v}$, whence if $\dot{v}=0$, then will $v = -a$. therefore $x = 2.302585a \times \text{Log.} \overline{y-x-a}$.

Ex. 42.

Suppose $a\dot{z} = z\dot{x} - x\dot{x}$. Assume $z = a + x + v$, then $\dot{z} = \dot{x} + \dot{v}$, the Values of z and $\dot{z}$ substituted in the given Equation give $a\dot{x} + a\dot{v} = a\dot{x} + x\dot{x} + v\dot{x} - x\dot{x}$, that is $a\dot{v} = v\dot{x}$, or $\dot{x} = \frac{a\dot{v}}{v}$, whence $x = a \times 2.302585 \times$ Log. v, therefore $z = a + v + a \times 2.302585 \times$ Log. v. Or $z = a + x +$ Number of the Logarithm $\frac{x}{2.30258a}$.

Ex. 43.

Suppose $\dot{z} = X^n x^m \dot{x}$, where X is the Hyperbolic Logarithm of x. Assume $z = \frac{X^n x^{m+1}}{m+1} + s$; this put into Fluxions there arises $\dot{z} = X^n x^m \dot{x} + \frac{nX^{n-1}\dot{X}x^{m+1}}{m+1} + \dot{s} = X^n x^m \dot{x}$, therefore $\dot{s} = \frac{-nX^{n-1}x^m\dot{x}}{m+1}$ (writing $\frac{\dot{x}}{x}$ for $\dot{X}$); Again assume $s = \frac{-nX^{n-1}x^{m+1}}{\overline{m+1}^2}$ $+ t$, this in Fluxions gives $\dot{t} = \frac{n.\overline{n-1}.X^{n-2}x^m\dot{x}}{\overline{m+1}^2}$

Again $t = \frac{n\overline{n-1}.X^{n-2}x^{m+1}}{\overline{m+1}^3} + u$. Whence $\dot{u} =$ $\frac{-n.\overline{n-1}.\overline{n-2}.X^{n-3}x^{m+1}\dot{x}}{\overline{m+1}^3}$, and $u =$

$$-\frac{n.\overline{n-1}.\overline{n-2}.X^{n-3}x^{m+1}}{\overline{m+1}^4} + w, \text{ \&c. Whence}$$

$$z = \frac{X^n x^{m+1}}{m+1} - \frac{nX^{n-1}x^{m+1}}{\overline{m+1}^2} + \frac{n.\overline{n-1}.X^{n-2}x^{m+1}}{\overline{m+1}^3}$$

$$-\frac{n.\overline{n-1}.\overline{n-2}.X^{n-3}x^{m+1}}{\overline{m+1}^4} + \&c.$$

And thus I have explain'd, as clearly as I could, the moſt *general Methods* of ſolving this *difficult Problem.* The Reader perhaps may think me too prolix on this Head: But it muſt be conſider'd that this is a Problem of the *greateſt Extent* and *Uſe* in the whole Practice of Fluxions; nay it contains almoſt the whole Science; and we cannot be too particular in treating on that which is the *Foundation* of the greateſt Part of the *Practice.* But after all we ſhall find it exceeding difficult in many Caſes to find the Fluents of Quantities, by any Methods hitherto known. And it is much to be wiſh'd that we had ſome eaſier Methods of finding Fluents, eſpecially of compound fluxionary Quantities, without the tedious Labour of reducing them to infinite Series, which in many Caſes converge ſo ſlow, and are ſo much compounded as to be in a manner uſeleſs. The following Problem is deſign'd to remedy this Difficulty in ſome particular Caſes.

PROP. XI.

To find the Fluent of a given Fluxion by the Table.

1. The following Table comprehends all Sorts of *Fluxions* and their correſpondent *Fluents*; not only ſuch as can be exactly had in *finite Terms*, and thoſe depending on the *Quadrature* of the *Conic Sections*, but alſo thoſe that can only be had by *infinite Series.*

2. Theſe

2. These Forms are all number'd in the first Column; the second Column contains the Fluxions, and the third gives the Fluent thereof. The 21st and all the following Forms relate to the *Transmutation* of Fluxions; here the Fluxions in the second and third Columns are equal, the second being transform'd into the third, and the Fluxion in the third Column always belongs to some of the foregoing Forms.

3. Here z, v, y express *variable Quantities*, and all the rest are *given* ones, which may represent any Quantities whatsoever affirmative or negative. But in the 6th, 8th, and 10th Forms, the Nature of them requires *negative* Quantities, and therefore they are written *negative*.

4. In the second Column are set down all the *necessary Conditions* relating to the *Signs*, *Indices*, &c. in each Form: likewise in what Cases the *Form* will *fail*, and when the Series will *terminate*. But the Fluents in Form 11, 12, 13, and 14 are design'd always to terminate, and are derived from the foregoing ten Forms, which ten Forms may therefore be called *original Forms*.

5. In these original Forms, though the *first* of the Fluents there given may in most Cases be sufficient, yet there are several *Varieties* both of *numerical* and *geometrical Fluents*; so that the most simple and elegant may always be chosen to suit any particular Case. It is sufficient to premise this concerning the Nature of these Forms.

A TABLE

A TABLE

Of FLUXIONS and FLUENTS.

L = 2.302585092994045684 &c.
N = ,017453292519943295 &c.

Forms.	*Fluxions.*	*Fluents.*
1	$z^{n-1}\dot{z}$ *This Form fails when* $n=0$.	$\frac{z^n}{n}$.
2	$z^{-1}\dot{z}=\dot{\phi}$	$\phi = L \times \text{Log}: z$. Or $\phi = \frac{Area}{RR}$ of the Hyperbola between the Assymptotes, whose inscrib'd Parallelogram is any Space RR, and Abscisa (taken in the Assymptote) Rz or z.
3	$\overline{\alpha+\beta z^n}\vert^{\mu} z^{n-1}\dot{z}$ *This Form fails when* $\mu=-1$.	$\frac{\overline{\alpha+\beta z^n}\vert^{\mu+1}}{\overline{\mu+1}.n\beta}$.
4	$\frac{z^{n-1}\dot{z}}{\alpha+\beta z^n} = \dot{\phi}$	$\phi = \frac{L}{n\beta} \times \text{Log. of } \overline{\alpha+\beta z^n}$. $\phi = \frac{Area}{n\beta RR}$ of an Hyperbola between the Assymptotes, whose inscribed Parallelogram is RR, Abscissa $\frac{\alpha}{\beta}+z^n$.

	Fluxions.	Fluents.
5	$\dfrac{z^{\frac{1}{2}\eta-1}\dot{z}}{\alpha+\beta z^{\eta}}=\dot{\varphi}$ *Here α and β are affirmative.*	$\varphi=\dfrac{2N}{\eta\sqrt{\alpha\beta}}\times$ *Degrees* in that Arch of a Circle whose Radius is 1, and natural Tangent $\sqrt{\dfrac{\beta z^{\eta}}{\alpha}}$. $\varphi=\dfrac{Arch}{\frac{1}{2}\eta R\sqrt{\alpha\beta}}$ of a Circle whose Radius is any Line R, and Tangent $R\sqrt{\dfrac{\beta z^{\eta}}{\alpha}}$. $\varphi=\dfrac{4\ Sectors}{\eta RR\sqrt{\alpha\beta}}$ of the Circle whose Radius is R, and Tangent $R\sqrt{\dfrac{\beta z^{\eta}}{\alpha}}$.
6	$\dfrac{z^{\frac{1}{2}\eta-1}\dot{z}}{\alpha-\beta z^{\eta}}=\dot{\varphi}$ *Here α is affirmative, and $-\beta$ negative.*	$\varphi=\dfrac{L}{\eta\sqrt{\alpha\beta}}\times$ Tab. Log. of $\dfrac{\sqrt{\alpha}+\sqrt{\beta z^{\eta}}}{\sqrt{\alpha}-\sqrt{\beta z^{\eta}}}$. $\varphi=\dfrac{L}{\eta\sqrt{\alpha\beta}}\times$ Log. of $\dfrac{\sqrt{\beta z^{\eta}}+\sqrt{\alpha}}{\sqrt{\beta z^{\eta}}-\sqrt{\alpha}}$. $\varphi=\dfrac{L}{\eta\sqrt{\alpha\beta}}\times$ Log. of $\dfrac{\alpha+\beta z^{\eta}+2\sqrt{\alpha\beta z^{\eta}}}{\alpha-\beta z^{\eta}}$. $\varphi=\dfrac{L}{\eta\sqrt{\alpha\beta}}\times$ Log. of $\dfrac{\alpha-\beta z^{\eta}}{\alpha+\beta z^{\eta}-2\sqrt{\alpha\beta z^{\eta}}}$. $\varphi=\dfrac{Area}{\eta RR\sqrt{\alpha\beta}}$ of the Hyperbola between the Assymptotes, whose inscribed Parallelogram is RR, and Abscissas (terminating this Area) $\sqrt{\alpha}+\sqrt{\beta z^{\eta}}$ and $\sqrt{\alpha}\backsim\sqrt{\beta z^{\eta}}$. $\varphi=\dfrac{4\ Sectors}{\eta RR\sqrt{\alpha\beta}}$ of a right angled Hyperbola, whose semitransverse is R, and Tangent at the Vertex $R\sqrt{\dfrac{\beta z^{\eta}}{\alpha}}$.

Forms.	Fluxions.
7	$\dfrac{z^{\frac{\delta}{\lambda}\eta - 1}\dot{z}}{\alpha + \beta z^{\eta}}$ *Here* α *and* β *are affirmative.* $\frac{\delta}{\lambda}$ *is an affirmative proper Fraction.*

Fluents.

$$\frac{1}{\eta\alpha^{\frac{\lambda-\delta}{\lambda}}\beta^{\frac{\delta}{\lambda}}} \times \text{ into } AL \times \text{Log.}\sqrt{1-2sx+xx} + BN \times P$$
$$+ CL \times \text{Log.}\sqrt{1-2tx+xx} + DN \times Q$$
$$+ EL \times \text{Log.}\sqrt{1-2ux+xx} + FN \times R \text{ \&c.}$$

K = an *Arch* of $\frac{180 \text{ Deg.}}{\lambda}$. $x = \overline{\frac{\beta z^n}{\alpha}}\Big|^{\frac{1}{\lambda}}$.

$a, b, c,$ &c = Sines ⎫ of 1K, 3K, 5K &c to 180°.
$s, t, u,$ &c = Cosines ⎭ Radius = 1.

P, Q, R &c. = *Degrees* of Arches whose Sines are
$\frac{ax}{\sqrt{1-2sx+xx}}$, $\frac{bx}{\sqrt{1-2tx+xx}}$, $\frac{cx}{\sqrt{1-2ux+xx}}$, &c.

CASES.

1. If $\frac{\delta}{\lambda} = \frac{1}{3}$, then $A = (-2s =) -1$. $B = 2a$. $C = 1$. $D = 0 = E = F$ &c.
2. If $\frac{\delta}{\lambda} = \frac{2}{3}$, $A = (2s =) 1$. $B = 2a$. $C = -1$. D, E, F &c. $= 0$.

---

3. If $\frac{\delta}{\lambda} = \frac{1}{4}$. $A = -2s$. $B = 2a$. $C = -2t$. $D = 2b$. ⎫
4. If $\frac{\delta}{\lambda} = \frac{3}{4}$. $A = 2s$. $B = 2a$. $C = 2t$. $D = 2b$. ⎭ E, F &c = 0.

---

5. If $\frac{\delta}{\lambda} = \frac{1}{5}$. $A = -2s$. $B = 2a$. $C = -2t$. $D = 2b$.
   $E = 1$. F &c $= 0$.
6. If $\frac{\delta}{\lambda} = \frac{2}{5}$. $A = 2t$. $B = 2b$. $C = 2s$. $D = -2a$.
   $E = -1$. F &c $= 0$.

---

7. If $\frac{\delta}{\lambda} = \frac{1}{6}$. $A = -2s$. $B = 2a$. $C = 0$ $D = 2b$.
   $E = -2u$. $F = 2c$. &c $= 0$.

&c.

Forms.	*Fluxions.*
8	$\dfrac{z^{\frac{\delta}{\lambda}\eta - 1}\dot{z}}{\alpha - \beta z^{\eta}}$ *Here* α *is affirmative, and* $-\beta$ *negative.* $\frac{\delta}{\lambda}$ *is an affirmative proper Fraction.*

Fluents.

$\dfrac{1}{n\beta^{\frac{\delta}{\lambda}}\alpha^{\frac{\lambda-\delta}{\lambda}}}$ X into $AL \times \text{Log.}\ \overline{1-x}. + BL \times \text{Log.}\ \sqrt{1-2sx+xx} + CN \times P + DL \times \text{Log.}\ \sqrt{1-2tx+xx} + EN \times Q + FL \times \text{Log.}\ \sqrt{1-2ux+xx} + GN \times R$ &c.

$K =$ An *Arch* of $\dfrac{180^\circ}{\lambda}$. $x = \overline{\dfrac{\beta z^n}{\alpha}}\Big|^{\frac{1}{\lambda}}$.

$a, b, c,$ &c $=$ Sines } of 2K, 4K, 6K, &c to 180°.
$s, t, u,$ &c $=$ Cosines } Radius $= 1$.

P, Q, R, &c. $=$ *Degrees* of Arches whose Sines are $\dfrac{ax}{\sqrt{1-2sx+xx}}, \dfrac{bx}{\sqrt{1-2tx+xx}}, \dfrac{cx}{\sqrt{1-2ux+xx}}$, &c.

CASES.

1. If $\dfrac{\delta}{\lambda} = \frac{1}{3}$. then $A = -1$. $B = -2s$. $C = 2a$.
2. If $\dfrac{\delta}{\lambda} = \frac{2}{3}$. $A = -1$. $B = -2s$. $C = -2a$.
   D, E &c $= 0$.

3. If $\dfrac{\delta}{\lambda} = \frac{1}{4}$. $A = -1$. $B = (-2s =)\ 0$. $C = 2a$. $D = 1$.
4. If $\dfrac{\delta}{\lambda} = \frac{3}{4}$. $A = -1$. $B = 0$. $C = -2a$. $D = 1$.
   E, F &c $= 0$.

5. If $\dfrac{\delta}{\lambda} = \frac{1}{5}$. $A = -1$. $B = -2s$. $C = 2a$. $D = -2t$. $E = 2b$.
6. If $\dfrac{\delta}{\lambda} = \frac{2}{5}$. $A = -1$. $B = -2t$. $C = 2b$. $D = -2s$. $E = -2a$.
   F, &c $= 0$.

7. If $\dfrac{\delta}{\lambda} = \frac{1}{6}$. $A = -1$. $B = -2s$. $C = 2a$. $D = -2t$.
   $E = 2b$. $F = 1$. G, &c. $= 0$.

&c.

Forms.	Fluxions.
9	$\dfrac{z^{\frac{1}{2}\eta-1}\dot{z}}{\sqrt{\alpha+\beta z^{\eta}}}=\dot{\phi}$ *Here β is affirmative.*

Fluents.

$\phi = \frac{2L}{n\sqrt{\beta}} \times \text{Log. of } \overline{\sqrt{\beta z^n} + \sqrt{\alpha + \beta z^n}}:$

$\phi = \frac{L}{n\sqrt{\beta}} \times \text{Log. of } \overline{\alpha + 2\beta z^n + 2\sqrt{\overline{\alpha + \beta z^n} \times \beta z^n}}:$

$\phi = \frac{-2L}{n\sqrt{\beta}} \times \text{Log. of } \overline{\sqrt{\alpha + \beta z^n} \backsim \sqrt{\beta z^n}}.$

$\phi = \frac{-L}{n\sqrt{\beta}} \times \text{Log. of } \overline{\alpha + 2\beta z^n - 2\sqrt{\alpha\beta z^n + \beta^2 z^{2n}}}.$

$\phi = \frac{4\,\textit{Sectors}}{n\text{RR}\sqrt{\beta}}$ of the right angled Hyperbola, whose Semitransverse is R, and Ordinate $R\sqrt{\frac{\beta z^n}{\alpha}}$.

$\phi = \frac{2}{n\text{RR}\sqrt{\beta}} \times$ *Sector* of the right Hyperbola, whose Semitransverse is R, and Ordinate $\frac{2R}{\alpha}\sqrt{\overline{\alpha + \beta z^n} \times \beta z^n}$.

$\phi = \frac{2}{n\text{RR}\sqrt{\beta}} \times$ *Sector* of the right Hyperbola, whose Semitransverse is R, and Ordinate $\frac{2R}{-\alpha}\sqrt{\overline{\alpha + \beta z^n} \times \beta z^n}$ when α stands for a negative Quantity.

$\phi = \frac{4\,\textit{Sectors}}{n\text{RR}\sqrt{\beta}}$ of the right Hyperbola, whose Semitransverse is R, and Ordinate $\frac{R}{\sqrt{-\alpha}} \times \sqrt{\alpha + \beta z^n}$, when α stands for a negative Quantity.

Forms.	*Fluxions.*
10	$\dfrac{z^{\frac{1}{2}\eta-1}\dot{z}}{\sqrt{\alpha-\beta z^{\eta}}}=\dot{\phi}$ *Here α is affirmative, and $-\beta$ is negative.*

Fluents.

$\varphi = \frac{2N}{\eta\sqrt{\beta}} \times$ *Degrees* in the Arch of a Circle whose Radius is 1, and natural Sine $\sqrt{\frac{\beta z^\eta}{\alpha}}$.

$\varphi = \frac{N}{\eta\sqrt{\beta}} \times$ *Degrees* in the Arch whose Radius is 1, and Sine $\frac{2}{\alpha}\sqrt{\overline{\alpha - \beta z^\eta} \times \beta z^\eta}$.

$\varphi = \frac{2}{\eta R\sqrt{\beta}} \times$ *Arch* whose Radius is R, and Sine $R\sqrt{\frac{\beta z^\eta}{\alpha}}$.

$\varphi = \frac{Arch}{\eta R\sqrt{\beta}}$, whose Radius is R, and Sine $\frac{2R}{\alpha}\sqrt{\overline{\alpha - \beta z^\eta} \times \beta z^\eta}$, or versed Sine $\frac{2R}{\alpha}\beta z^\eta$.

$\varphi = \frac{4}{\eta RR\sqrt{\beta}} \times$ *Sector* of a Circle, whose Radius is any Line R, and Sine $R\sqrt{\frac{\beta z^\eta}{\alpha}}$.

$\varphi = \frac{2\ Sectors}{\eta RR\sqrt{\beta}}$ of the Circle whose Radius is R, and Sine $\frac{2R}{\alpha}\sqrt{\alpha\beta z^\eta - \beta^2 z^{2\eta}}$.

Forms.	*Fluxions.*
11	$\overline{\alpha+\beta z^{n}}^{\mu} z^{\pi+\lambda n}\dot{z}$ *Here* λ *is any affirmative whole Number.* N. B. *This Form fails when* $\frac{\pi+1}{n}+\mu$ *is any negative whole Number from* -1 *to* $-\lambda$ *inclusive.*
12	$\overline{\alpha+\beta z^{n}}^{\mu} z^{\pi-\lambda n}\dot{z}$ *Here* λ *is any affirmative whole Number.* N. B. *This Form fails when* $\frac{\pi+1}{n}$ *is any affirmative whole Number from* 1 *to* λ *inclusively.*

Fluents.

$$\frac{\overline{-\alpha}^{\lambda}\cdot\overline{\varepsilon-\eta}\cdot\overline{\varepsilon-2\eta}\cdot\overline{\varepsilon-3\eta}\cdot\ \&c\ \ldots\ \text{till}\ \overline{\varepsilon-\lambda\eta}}{\beta^{\lambda}\cdot\overline{\delta-\eta}\cdot\overline{\delta-2\eta}\cdot\overline{\delta-3\eta}\cdot\ \&c\ \ldots\ \text{till}\ \overline{\delta-\lambda\eta}}\,\varphi$$

$$+\frac{z^{\varepsilon-\eta}\gamma}{\overline{\delta-\eta}\cdot\beta}-\frac{\overline{\varepsilon-\eta}\cdot\alpha}{\overline{\delta-2\eta}\cdot\beta z^{\eta}}A-\frac{\overline{\varepsilon-2\eta}\cdot\alpha}{\overline{\delta-3\eta}\cdot\beta z^{\eta}}B$$

$$-\frac{\overline{\varepsilon-3\eta}\cdot\alpha}{\overline{\delta-4\eta}\cdot\beta z^{\eta}}C-\ \&c\ \text{till}\ -\frac{\overline{\varepsilon+\eta-\lambda\eta}\cdot\alpha}{\overline{\delta-\lambda\eta}\cdot\beta z^{\eta}}F.$$

$\varphi =$ Fluent of $\overline{\alpha+\beta z^{\eta}}^{\mu}\, z^{\pi}\dot{z}$.

$\gamma = \overline{\alpha+\beta z^{\eta}}^{\mu+1}$.

$\varepsilon = \pi + 1 + \lambda\eta$.

$\delta = \varepsilon + \eta + \mu\eta$.

A, B, C *&c*, each preceding Term with its Sign.

$$\frac{\beta^{\lambda}\cdot\overline{\delta+\eta}\cdot\overline{\delta+2\eta}\cdot\overline{\delta+3\eta}\cdot\ \&c\ \ldots\ \text{till}\ \overline{\delta+\lambda\eta}}{\overline{-\alpha}^{\lambda}\cdot\overline{\varepsilon+\eta}\cdot\overline{\varepsilon+2\eta}\cdot\overline{\varepsilon+3\eta}\cdot\ \&c\ \ldots\ \text{till}\ \overline{\varepsilon+\lambda\eta}}\,\varphi$$

$$+\frac{z^{\varepsilon+\eta}\gamma}{\overline{\varepsilon+\eta}\cdot\alpha}-\frac{\overline{\delta+\eta}\ \beta z^{\eta}}{\overline{\varepsilon+2\eta}\cdot\alpha}A-\frac{\overline{\delta+2\eta}\cdot\beta z^{\eta}}{\overline{\varepsilon+3\eta}\cdot\alpha}B$$

$$-\frac{\overline{\delta+3\eta}\cdot\beta z^{\eta}}{\overline{\varepsilon+4\eta}\cdot\alpha}C-\ \&c\ \text{till}\ -\frac{\overline{\delta-\eta+\lambda\eta}\cdot\beta z^{\eta}}{\overline{\varepsilon+\lambda\eta}\cdot\alpha}F.$$

$\varphi =$ Fluent of $\overline{\alpha+\beta z^{\eta}}^{\mu}\, z^{\pi}\dot{z}$.

$\gamma = \overline{\alpha+\beta z^{\eta}}^{\mu+1}$.

$\varepsilon = \pi + 1 - \lambda\eta - \eta$.

$\delta = \varepsilon + \mu\eta + \eta$.

A, B, C *&c*, each preceding Term with its Sign.

Forms.	Fluxions.
13	$\overline{\alpha + \beta z^{\eta}}\vert^{\mu+\tau} z^{\pi} \dot{z}$ *Here* τ *is any affirmative whole Number.* N. B. *This Form fails when* $\frac{\pi+1}{\eta} + \mu$ *is any negative whole Number from* -1 *to* $-\tau$ *inclusively.*
14	$\overline{\alpha + \beta z^{\eta}}\vert^{\mu-\tau} z^{\pi} \dot{z}$ *Here* τ *is any affirmative whole Number.* N. B. *This Form will fail when* $\mu + 1$ *is any affirmative whole Number, from* 1 *to* τ *inclusive.*

Fluents.

$$\frac{\eta^{\tau}\alpha^{\tau}\,.\,\overline{\mu+1}\,.\,\overline{\mu+2}\,.\,\overline{\mu+3}.\ \&c.\ \ldots\ \text{till}\ \overline{\mu+\tau}}{\overline{\delta+\eta}\,.\,\overline{\delta+2\eta}\,.\,\overline{\delta+3\eta}.\ \&c\ \ldots\ \text{till}\ \overline{\delta+\tau\eta}}\varphi$$

$$+\frac{z^{\pi+1}\gamma^{\mu+\tau}}{\delta+\tau\eta}+\frac{\overline{\mu+\tau}.\eta\alpha}{\overline{\delta+\tau\eta-\eta}.\gamma}A+\frac{\overline{\mu+\tau-1}.\eta\alpha}{\overline{\delta+\tau\eta-2\eta}.\gamma}B$$

$$+\frac{\overline{\mu+\tau-2}.\eta\alpha}{\overline{\delta+\tau\eta-3\eta}.\gamma}C+\&c.\ \text{untill}\ \frac{\overline{\mu+2}.\eta\alpha}{\overline{\delta+\eta}.\gamma}F.$$

$\varphi =$ Fluent of $\overline{\alpha+\beta z^{n}}^{\mu}z^{\pi}\dot{z}$.

$\gamma = \alpha + \beta z^{n}$.

$\delta = \pi + 1 + \mu\eta$.

A, B, C, &c, each preceding Term with its Sign.

$$\frac{\delta\,.\,\overline{\delta-\eta}\,.\,\overline{\delta-2\eta}\,.\,\overline{\delta-3\eta}.\ \&c\ \ldots\ \text{till}\ \overline{\delta+\eta-\tau\eta}}{\eta^{\tau}\alpha^{\tau}\mu.\overline{\mu-1}\,.\,\overline{\mu-2}\,.\,\overline{\mu-3}.\ \&c\ \ldots\ \text{till}\ \overline{\mu+1-\tau}}\varphi$$

$$-\frac{z^{\pi+1}\gamma^{\mu-\tau+1}}{\overline{\mu-\tau+1}.\eta\alpha}+\frac{\overline{\delta-\tau\eta+\eta}.\gamma}{\overline{\mu-\tau+2}.\eta\alpha}A+\frac{\overline{\delta-\tau\eta+2\eta}.\gamma}{\overline{\mu-\tau+3}.\eta\alpha}B$$

$$+\frac{\overline{\delta-\tau\eta+3\eta}.\gamma}{\overline{\mu-\tau+4}.\eta\alpha}C+\&c\ \text{till}\ \frac{\overline{\delta-\eta}.\gamma}{\mu\eta\alpha}F.$$

$\varphi =$ Fluent of $\overline{\alpha+\beta z^{n}}^{\mu}z^{\pi}\dot{z}$.

$\gamma = \alpha + \beta z^{n}$.

$\delta = \pi + 1 + \mu\eta$.

A, B, C &c, each preceding Term with its Sign.

Forms.	Fluxions.
15	$\overline{\alpha + \beta z^{\eta}}^{\mu} z^{\pi} \dot{z}$ *This Series will terminate when* $\frac{\pi + 1}{\eta} + \mu$ *is any negative whole Number.* N. B. *This Form fails when* $\frac{\pi + 1}{\eta}$ *is any negative whole Number whatsoever.*
16	$\overline{\alpha + \beta z^{\eta}}^{\mu} z^{\pi} \dot{z}$ *This Series will terminate only when* μ *is any affirmative whole Number.*
17	$\overline{\alpha + \beta z^{\eta}}^{\mu} z^{\pi} \dot{z} \times$ into $e + fz^{n} + gz^{2n} + hz^{3n}$ &c. *Here* μ *and* π *must each be greater than* -1. *Also* z *and* $\overline{\alpha + \beta z^{\eta}}\vert^{\mu+1}$ *must be one of them* $= 0$ *at the beginning of the Fluent, the other* $= 0$ *at the end of it.* *This Series will terminate when* $\frac{\pi + 1}{\eta}$ *is any negative Number whatever.*

Fluents.

$$\frac{z^{\epsilon}}{\epsilon\alpha} - \frac{\delta+\eta}{\epsilon+\eta}Aq - \frac{\delta+2\eta}{\epsilon+2\eta}Bq - \frac{\delta+3\eta}{\epsilon+3\eta}Cq$$
$$- \frac{\delta+4\eta}{\epsilon+4\eta}Dq - \&c \times \text{ into } \overline{\alpha+\beta z^{\eta}}|^{\mu+1}.$$

$\epsilon = \pi + 1.$

$\delta = \epsilon + \mu\eta.$

$q = \frac{\beta z^{n}}{\alpha}.$

A, B, C &c, each preceeding Term with its Sign.

$$\frac{\alpha^{\mu}z^{\pi+1}}{\pi+1} + \frac{\mu Aq}{\pi+1+\eta} + \frac{\frac{\mu-1}{2}Bq}{\pi+1+2\eta} + \frac{\frac{\mu-2}{3}Cq}{\pi+1+3\eta}$$
$$+ \frac{\frac{\mu-3}{4}Dq}{\pi+1+4\eta} + \frac{\frac{\mu-4}{5}Eq}{\pi+1+5\eta} + \&c.$$

$q = \frac{\beta z^{n}}{\alpha}.$

A, B, C &c are the Numerators of each preceding Term with its Sign.

$$\phi \times : e - \frac{f\epsilon\alpha}{\delta+\eta.\beta} - \frac{g.\overline{\epsilon+\eta}.\alpha}{f.\overline{\delta+2\eta}.\beta}A - \frac{h.\overline{\epsilon+2\eta}.\alpha}{g.\overline{\delta+3\eta}.\beta}B$$
$$- \frac{k.\overline{\epsilon+3\eta}.\alpha}{h.\overline{\delta+4\eta}.\beta}C, \&c.$$

$\epsilon = \pi + 1.$

$\delta = \pi + 1 + \mu\eta.$

$\phi =$ Fluent of $\overline{\alpha + \beta z^{n}}|^{\mu} z^{\pi}\dot{z}.$

A, B, C &c, each preceding Term with its Sign.

Forms.	*Fluxions.*
18	$\overline{e+fz^{n}+gz^{2n}+hz^{3n}\ \&c} \times$ into $\overline{\alpha+\beta z^{n}+\gamma z^{2n}+\delta z^{3n}\ \&c}^{\,\mu}\ z^{\pi}\dot{z}$. N. B. *This Form will fail when* $\frac{\pi+1}{n}$ *is any negative whole Number whatever.*
19	$\overline{mz\dot{y}+ny\dot{z}} \times z^{n-1}y^{m-1}$ or $\overline{\frac{m\dot{y}}{y}+\frac{n\dot{z}}{z}} \times z^{n}y^{m}$
20	$\overline{rzy\dot{v}+mzv\dot{y}+nyv\dot{z}} \times$ into $z^{n-1}\ y^{m-1}\ v^{r-1}$.

Fluents.

$$+\frac{e}{p\alpha}$$

$$+\frac{f\left.\begin{matrix}-p\\-mn\end{matrix}\right\}\beta A}{\overline{p+n}\cdot\alpha}z^{n}$$

$$+\frac{g\left.\begin{matrix}-p\\-2mn\end{matrix}\right\}\gamma A\left.\begin{matrix}-\overline{p+n}\\-mn\end{matrix}\right\}\beta B}{\overline{p+2n}\cdot\alpha}z^{2n}$$

$$+\frac{h\left.\begin{matrix}-p\\-3mn\end{matrix}\right\}\delta A\left.\begin{matrix}-\overline{p+n}\\-2mn\end{matrix}\right\}\gamma B\left.\begin{matrix}-\overline{p+2n}\\-mn\end{matrix}\right\}\beta C}{\overline{p+3n}\cdot\alpha}z^{3n}$$

$$+\frac{i\left.\begin{matrix}-p\\-4mn\end{matrix}\right\}\varepsilon A\left.\begin{matrix}-\overline{p+n}\\-3mn\end{matrix}\right\}\delta B\left.\begin{matrix}-\overline{p+2n}\\-2mn\end{matrix}\right\}\gamma C\left.\begin{matrix}-\overline{p+3n}\\-mn\end{matrix}\right\}\beta D}{\overline{p+4n}\cdot\alpha}z^{4n}$$

$+$ &c X into $z^{p}\times\overline{\alpha+\beta z^{n}+\gamma z^{2n}+\delta z^{3n}\ \&c.}^{\mu+1}$

$p=\pi+1.$

$m=\mu+1.$

A, B, C &c, the Coefficients of the 1st, 2d, 3d &c. Terms respectively.

$z^{n}y^{m}.$

$z^{n}y^{m}v^{r}.$

Forms.	*Given Fluxions*
21	$\overline{e+fz^{\theta}}^{m} z^{\lambda\theta-1}\dot{z}$ λ *is any whole number whatever.*
22	$\dfrac{z^{\lambda\theta-1}\dot{z}}{\overline{e+fz^{\theta}} \times \overline{g+hz^{\theta}}}$
23	$\overline{e+fz^{\theta}}^{m} \times \overline{g+hz^{\theta}}^{r} z^{\lambda\theta-1}\dot{z}$ *Here* λ *is any whole Number greater than* o. r *is half of any whole Number whatever.*
24	$\dfrac{\overline{e+fz^{\theta}}^{m} z^{\lambda\theta-1}\dot{z}}{\overline{g+hz^{\theta}}^{m+r}}$
	Here λ *is any whole Number* ~~*greater than* o.~~ r *is any positive whole Number whatever (except when both* λ *and* r *are* o*).*

Transform'd into others.
$\overline{v-e}^{\lambda-1} \times \frac{v^{m}\dot{v}}{\theta f^{\lambda}}$. $v=e+fz^{\theta}$.
$\frac{fz^{\lambda\theta-1}\dot{z}}{p.\overline{e+fz^{\theta}}} - \frac{hz^{\lambda\theta-1}\dot{z}}{p.\overline{g+hz^{\theta}}}$. $p=fg-eh$.
$\overline{v-e}^{\lambda-1} \times \overline{p+\frac{hv}{f}}\Big\vert^{r} \times \frac{v^{m}\dot{v}}{\theta f^{\lambda}}$. $p=g-\frac{eh}{f}$. $v=e+fz^{\theta}$.
$\overline{gv-e}^{\lambda-1} \times \frac{\overline{f-hv}^{r-\lambda-1}\, v^{m}\dot{v}}{\theta p^{r-1}}$. $p=fg-eh$. $v=\frac{e+fz^{\theta}}{g+hz^{\theta}}$.

Forms.	Fluxions
25	$\dfrac{\overline{e+fz^{\theta}}^{m} z^{\theta-1}\dot{z}}{\overline{k+lz^{\theta}}\,.\,\overline{g+hz^{\theta}}^{m+r}}$ *Here r is any affirmative whole Number whatsoever.*
26	$\dfrac{z^{\lambda\theta-1}\dot{z}}{e+fz^{\theta}+gz^{2\theta}}$ N.B. *This Form fails when 4eg is greater than ff.*
27	$\overline{e+fz^{\theta}+gz^{2\theta}}^{m}\; z^{\lambda\theta-1}\dot{z}$ *λ is any positive whole Number greater than* 0. *m is the half of any whole Number.*
28	$\dfrac{z^{\lambda\theta-1}\dot{z}}{e+fz^{\theta}+gz^{2\theta}}$ *Here e and g are affirmative.* *λ is the half of any affirmative whole Number greater than* 1. N.B. *This Form fails when ff is greater than 4eg.*

Transform'd.

$$\frac{\overline{f-hv}^{r}}{\theta l b^{r}} \times \frac{av^{m}\dot{v}}{d+av} + \frac{\overline{f-hv}^{r-1} hv^{m}\dot{v}}{\theta l b^{r}}.$$

$a = gl - kh.$
$b = fg - eh.$
$d = fk - el.$
$v = \dfrac{e+fz^{\theta}}{g+hz^{\theta}}.$

$$\frac{g}{p} \times \frac{z^{\lambda\theta-1}\dot{z}}{\frac{f-p}{2}+gz^{\theta}} - \frac{g}{p} \times \frac{z^{\lambda\theta-1}\dot{z}}{\frac{f+p}{2}+gz^{\theta}}.$$

$p = \sqrt{ff-4eg}.$

$$\frac{\overline{v-\frac{1}{2}f}^{\lambda-1}}{\theta g^{\lambda}} \times \overline{p+\frac{v^{2}}{g}}\Big|^{m} \dot{v}.$$

$p = e - \dfrac{ff}{4g}.$
$v = \frac{1}{2}f + gz^{\theta}.$

$$\frac{\overline{v+\frac{1}{2}s}^{2\lambda-2}}{\theta s g^{2\lambda-3}} \times \frac{\dot{v}}{p+v^{2}} - \frac{\overline{V-\frac{1}{2}s}^{2\lambda-2}}{\theta s g^{2\lambda-3}} \times \frac{\dot{V}}{p+V^{2}}.$$

$p = \frac{1}{4}fg + \frac{1}{2}g\sqrt{eg}.$
$s = \sqrt{2g\sqrt{eg}-fg}.$
$v = -\frac{1}{2}s + gz^{\frac{1}{2}\theta}.$
$V = +\frac{1}{2}s + gz^{\frac{1}{2}\theta}.$

Forms.	Fluxions
29	$\dfrac{z^{\lambda\theta-1}\dot{z}}{\overline{h+kz^{\theta}}\times\overline{e+fz^{\theta}+gz^{2\theta}}}$ N. B. *This Form fails when* 4*eg is greater than ff*.
30	$\dfrac{z^{\lambda\theta-1}\dot{z}}{\overline{k+lz^{\theta}}\times\overline{e+fz^{\theta}+gz^{2\theta}}}$ *Here* λ *is any affirmative whole Number greater than* 0.
31	$\dfrac{z^{\lambda\theta-1}\dot{z}}{\overline{k+lz^{\theta}}\times\overline{e+fz^{\theta}+gz^{2\theta}}}$ *Here e and g are affirmative.* λ *is the half of any affirmative whole Number greater than* 2. N. B. *This Form fails when ff is greater than* 4*eg*.

Transform'd.

$$\frac{gkkz^{\lambda\theta-1}\dot{z}}{cd.\overline{b+kz^{\theta}}}+\frac{ggz^{\lambda\theta-1}\dot{z}}{pd.\overline{b+gz^{\theta}}}-\frac{ggz^{\lambda\theta-1}\dot{z}}{pc.\overline{a+gz^{\theta}}}.$$

$p=\sqrt{ff-4eg}$.
$a=\frac{f-p}{2}$.
$b=\frac{f+p}{2}$.
$c=ak-gh$.
$d=bk-gh$.

$$\frac{llz^{\lambda\theta-1}\dot{z}}{t.\overline{k+lz^{\theta}}}+\frac{\overline{v-\frac{1}{2}f}\big|^{\lambda-1}}{\theta tg^{\lambda-1}}\times\frac{a\dot{v}-lv\dot{v}}{p+v^{2}}.$$

$a=gk-\frac{1}{2}lf$.
$p=eg-\frac{1}{4}ff$.
$t=ell-fkl+gkk$.
$v=\frac{1}{2}f+gz^{\theta}$.

$$-\frac{lkz^{\lambda\theta-\theta-1}\dot{z}}{t.\overline{k+lz^{\theta}}}+\frac{\overline{v+\frac{1}{2}s}\big|^{2\lambda-3}}{2\theta stg^{2\lambda-3}}\times\frac{-2qv\dot{v}+rs\dot{v}}{p+v^{2}}$$

$$+\frac{\overline{V-\frac{1}{2}s}\big|^{2\lambda-3}}{2\theta stg^{2\lambda-3}}\times\frac{2qV\dot{V}+rs\dot{V}}{p+V^{2}}.$$

$p=\frac{1}{2}g\sqrt{eg}+\frac{1}{4}fg$.
$q=l\sqrt{eg}-gk$.
$r=l\sqrt{eg}+gk$.
$s=\sqrt{2g\sqrt{eg}-fg}$.
$t=ell-fkl+gkk$.
$v=-\frac{1}{2}s+gz^{\frac{1}{2}\theta}$.
$V=+\frac{1}{2}s+gz^{\frac{1}{2}\theta}$.

Forms.	Fluxions
32	$$\frac{z^{\theta-1}\dot{z}}{\overline{k+lz^{\theta}}\,.\,\overline{e+fz^{\theta}+gz^{2\theta}}\big\vert^{m}}$$ *Here m is half of any positive whole Number.*
33	$$\frac{z^{\lambda\theta-1}\dot{z}\,.\,\overline{e+fz^{\theta}}\vert^{m}}{h+kz^{\theta}+lz^{2\theta}}$$ *Here λ is any affirmative whole Number greater than* 0. N. B. *This Form fails when 4hl is greater than kk.*

Transform'd.

$$\frac{\overline{lv-\frac{1}{2}b}|^{2m-1}}{-n} \times \frac{\dot{v}}{\overline{ap+avv}|^{m}}.$$

$$a = ell - fkl + gkk.$$
$$b = fl - 2gk.$$
$$p = eg - \tfrac{1}{4}ff.$$
$$v = \frac{el - \frac{1}{2}fk + \frac{1}{2}bz^{\theta}}{k + lz^{\theta}}.$$

$$\frac{l \times \overline{v-e}|^{\lambda-1}}{\theta pf^{\lambda-1}} \times \frac{v^{m}\dot{v}}{\frac{k-p}{2}f - le + lv}$$

$$-\frac{l \times \overline{v-e}|^{\lambda-1}}{\theta pf^{\lambda-1}} \times \frac{v^{m}\dot{v}}{\frac{k+p}{2}f - le + lv}.$$

$$p = \sqrt{kk - 4hl}.$$
$$v = e + fz^{\theta}.$$

Forms.	Fluxions
34	$$\frac{z^{\lambda\theta-1}\dot{z} \times \overline{e+fz^{\theta}}^{m}}{h+kz^{\theta}+lz^{2\theta}}$$ *Here h and l are affirmative.* λ *is any positive whole Number greater than* 0. *m is the half of any affirmative odd Number.* N.B. *This Form fails when kk is greater than 4hl, or when a is negative.*
35	$$\frac{z^{\lambda\theta-1}\dot{z}}{\overline{h+kz^{\theta}+lz^{2\theta}} \times \overline{e+fz^{\theta}}^{m}}$$ *Here h and l are affirmative.* λ *is any positive whole Number less than* $m+1$. *m is the half of any positive odd Number.* N. B. *This Form will fail when kk is greater than 4hl, or when a is negative.*

Transform'd.

$$\frac{\overline{vv - sv + q}^{\lambda-1} \times \overline{v + \frac{1}{2}s}^{2m}}{\theta s f^{\lambda-2}\, l^{2\lambda+2m-3}} \times \frac{\dot{v}}{p + vv}$$

$$- \frac{\overline{VV + sV + q}^{\lambda-1} \times \overline{V - \frac{1}{2}s}^{2m}}{\theta s f^{\lambda-2}\, l^{2\lambda+2m-3}} \times \frac{\dot{V}}{p + VV}.$$

$a = ffh - efk + eel.$
$b = fk - 2el.$
$s = \sqrt{2l\sqrt{al} - bl}.$
$p = l\sqrt{al} - \frac{1}{4}ss.$
$q = \frac{1}{4}ss - ell.$
$v = -\frac{1}{2}s + l\sqrt{e + fz^{\theta}}.$
$V = +\frac{1}{2}s + l\sqrt{e + fz^{\theta}}.$

$$+ \frac{\overline{-eVV + esV + q}^{\lambda-1} \times \overline{V - \frac{1}{2}s}^{2m-2\lambda+2}}{\theta s f^{\lambda-2} a^{m} l^{m}} \times \frac{l\dot{V}}{p + V^2}$$

$$- \frac{\overline{-evv - esv + q}^{\lambda-1} \times \overline{v + \frac{1}{2}s}^{2m-2\lambda+2}}{\theta s f^{\lambda-2} a^{m} l^{m}} \times \frac{l\dot{v}}{p + v^2}.$$

$a = ffh - efk + eel.$
$b = fk - 2el.$
$s = \sqrt{2l\sqrt{al} - bl}.$
$p = l\sqrt{al} - \frac{1}{4}ss.$
$q = al - \frac{1}{4}sse.$
$v = -\frac{1}{2}s + \sqrt{\frac{al}{e + fz^{\theta}}}.$
$V = +\frac{1}{2}s + \sqrt{\frac{al}{e + fz^{\theta}}}.$

The Use of the foregoing Table of Fluxions.

1. WHEN it is required to find the Fluent of a given Fluxion, by this Table; it muſt firſt be found out to what *Form* it belongs. To this End *ſubſtitute* η or θ (as the Caſe requires) for the *numeral Index*, and then it may be compared with the ſeveral Forms in the Table. Or the given Fluxion may (and often muſt) be *reduced* to another Expreſſion, by dividing it by the higheſt Power in the Radical, (that is by taking the higheſt Power of the variable Quantity out of the Radical) and affecting the other Quantities with it; by which Means the Fluxion will acquire a new Form and Expreſſion, for the Sign of the Index will be changed, *(and this I call reducing the Index)*. Then ſubſtitute η or θ for the numeral Index, and ſee what Form in the Table it will agree to.

2. If the Fluxion, then, is a Binomial, you muſt firſt try if the Fluent of either Expreſſion can be had in *finite Terms* by Form 3d or 15th, (which may always be known by the Notes in the ſecond Column); if it cannot, then try whether it can be had in finite Terms by any other of the Forms for Binomials, which will be eaſy by comparing the Indices. If it fall under the 11th, 12th, 13th, or 14th Form, there will be required two (or perhaps three) Operations, whereof the *firſt* is always for the Fluent of the *original Fluxion*, and is to be found by ſome of the firſt ten Forms. But if it cannot be found in finite Terms by theſe Forms, then the *laſt Recourſe* is to the 15th, 16th, 17th, or 18th Form, as beſt ſuits the Caſe.

3. Having found to what Form (or Forms) the Fluxion belongs, you have no more to do but only to *write* the reſpective *Values* of the *general Quantities* in the Fluent belonging to that *Form*, and multiply the

the *Refult* by fuch given Quantities as the given *Fluxion* was multiplied into. and you have the *Fluent*.

4. And in compound Binomials, or Trinomials, fuch as belong to the 21ft and following Forms, fince thefe are transformed into fingle binomial Fluxions, ftanding in the third Column; therefore you muft proceed with thefe according to the foregoing Directions for Binomials; and the Fluents of thefe (fingle) Binomials being found, will be the Fluents of the Trinomial or compound Binomial Fluxions in the fecond Column, refpectively.

5. But there are feveral Sorts of compound Fluxions which cannot be refolved without fome further Reduction. Now there are thefe two other Ways of reducing a given Fluxion to a different Form or Expreffion. The firft is actually to multiply by any Power of the Quantity under the Vinculum (in any Binomial or Trinomial Surd) and then leffen the Index of that Surd by the fame Power if it is in the Numerator, or increafe it in the Denominator: This alters the Power of the furd Quantity. The other way is actually to divide by the Quantity under the Vinculum, and then leffen the Index of that Surd by 1 if it be in the Denominator, or increafe it by 1 if it is in the Numerator: This alters the Dimenfion of the fimple variable Quantity. And in both you will have the more Terms according as you multiply by a greater Power, or continue your Divifion the further. But the laft Term only will be of a like Form with the given Fluxion, differing only in the Index of the Surd, or of the variable Quantity.

6. Therefore in any Fluxion, but efpecially in compound Binomials and Trinomials, if there be a Surd in the Numerator you may *leffen* its Dimenfion at Pleafure, (or take it quite out of the Numerator) if you *multiply* by *fome Power* of the Quantity under the Vinculum, and *leffen* its Index by the *fame* Power.

7. Alſo in any given Fluxion, if the Index of the ſimple variable Quantity be too high, (as 2θ, 3θ, $r\theta$, *&c*) or when it is too low or Negative, (as -1, $-\theta$, -2θ, *&c*) it may be altered at Pleaſure, by *dividing* by ſome rational compound Quantity in the Denominator if there is any, or elſe by the Quantity under the Vinculum, and *ſubſtracting* 1 from its Index, if it is in the *Denominator*, or *adding* 1 in the *Numerator*; and continuing the Diviſion to a proper Number of Places.

By theſe Operations the given Fluxion is reduced to ſeveral Terms, all which (except the laſt) muſt be ſtill farther reduced if there be Occaſion by repeating the ſame Method, till at laſt all the reſulting Terms will fall under ſome or other of the Forms in the Table; Examples whereof will follow afterwards. By theſe Sorts of Reduction a given Fluxion is prepared for a Solution, when it does not fall directly at firſt under any of the Forms.

8. In Form 16th, when the *Denominator* of any Term happens to be o, that particular Term muſt be found thus; take the *known Part* of the Numerator, (that is leaving out the capital Letter and the Powers of z) and *multiply* it into 2. 3025851 *Log.* z, and you have that Term. And the ſaid known Part of the Numerator will be the Value of the capital Letter in the next following Term.

9. Though theſe Forms contain Variety of logarithmic Fluents, yet each of them may be changed ſeveral Ways into different Quantities. Thus, when a Logarithm is in the Fluent, you may multiply (the *Number* whoſe Log. is there) and then divide it by ſome compound Quantity, which you ſee will make it ſimpler: Or you may ſquare it and take half the Logarithm: Or you may extract the ſquare Root and take double the Logarithm: Or you may multiply or divide it by any given Quantity: Or make the Numerator and Denominator change Places, and then

then change the Sign of the Logarithm, &c. And thus you may always find the simplest Expression for the Logarithmic Quantity.

10. And if you have an untractable Fluxion that will answer to none of the Forms, it may sometimes be *transformed* into others, by Prop. IX, which may then be resolved by the Table.

The following Examples will make the Process very plain.

Example 1.

To find the Fluent of $\frac{x\dot{x}}{xx-aa}$.

Here $\eta=2$, by which expunging the numeral Index, the Fluxion will be reduced to $\frac{x^{\eta-1}\dot{x}}{-aa+x^{\eta}}$, which is a Fluxion belonging to the 4th Form. Therefore $\alpha=-aa$, $\beta=1$, $z=x$, and $\frac{2.3025}{\eta\beta}\times$ Log. of $\overline{\alpha+\beta z^{\eta}}=\frac{2.3025}{2}\times$ Log. $\overline{xx-aa}=2.3025851\times$ Log. $\sqrt{x^2-a^2}$, the Fluent required.

Ex. 2.

Let the Fluxion $\frac{r\dot{x}}{\sqrt{bb+xx}}$ *be given.*

Here $\eta=2$; whence the Fluxion will be reduced to $\frac{rx^{\frac{1}{2}\eta-1}\dot{x}}{\sqrt{bb+x^{\eta}}}$, a Fluxion of the 9th Form; whence $\alpha=bb$. $\beta=1$. $z=x$, and $\phi=2.30258$ Log: $\overline{x+\sqrt{bb+xx}}$: $=$ Fluent of $\frac{\dot{x}}{\sqrt{bb+xx}}$; therefore the Fluent of $\frac{r\dot{x}}{\sqrt{bb+xx}}=2.302585\,r\times$ Log: $\overline{x+\sqrt{bb+xx}}$:

Ex. 3.

Ex. 3.

To find the Fluent of $\dfrac{bx^{\frac{1}{3}}\dot{x}}{\overline{c-ax^{\frac{2}{3}}}|^{\frac{3}{5}}}$.

Here $\eta=\frac{2}{3}$, and the Fluxion becomes $\overline{c-ax^{\eta}}|^{-\frac{3}{5}}\times bx^{2\eta-1}\dot{x}$, which belongs to Form the 3d and 11th. Or rather thus, by taking $x^{\frac{2}{3}}$ out of the Radical, the Fluxion becomes $\overline{-a+cx^{-\frac{2}{3}}}|^{-\frac{3}{5}}\times bx^{-\frac{1}{15}}\dot{x}$, which belongs to Form 15th. Here $\alpha=-a$, $\beta=c$, $z=x$, $\eta=-\frac{2}{3}$, $\mu=-\frac{3}{5}$, $\pi=-\frac{1}{15}$, $\epsilon=\frac{14}{15}$, $\delta=\frac{4}{3}$. Then since $\frac{\pi+1}{\eta}+\mu=-2$ a negative whole Number, therefore the Fluent will be had in finite Terms, and is $=b\times:-\frac{15x^{\frac{14}{15}}}{14a}-\frac{75cx^{\frac{4}{15}}}{28aa}:\times\overline{-a+cx^{-\frac{2}{3}}}|^{\frac{2}{5}}=$ $-\frac{30abx^{\frac{2}{3}}+75bc}{28aa}\times\overline{c-ax^{\frac{2}{3}}}|^{\frac{2}{5}}$, the Fluent required.

Ex. 4.

To find the Fluent of $\dfrac{b\dot{z}}{z\sqrt{zz-cc}}$.

This belongs to Form the 12th, having by Form the 3d the Fluent of $\frac{bz\dot{z}}{\sqrt{zz-cc}}$; in which Case, $\pi=1$, $\eta=2$, and $-1=1-2=\pi-\eta$, and the given Fluxion becomes $\frac{bz^{\pi-\eta}}{\sqrt{-c^2+z^{\eta}}}$. But since $\frac{\pi+1}{\eta}=1$, therefore the Fluent cannot be had by Form 12th. Wherefore I reduce it to $-\frac{bz^{-2}\dot{z}}{\sqrt{1-ccz^{-2}}}$, which is a Fluxion of the 10th Form, where $\eta=-2$, $\alpha=1$, $\beta=-cc$; and the Fluent $=-\frac{b}{c}\times$,017453 Degrees of the Arch whose Sine is $\frac{c}{z}$.

Ex. 5.

Ex. 5.

Let the Fluent of $\frac{a\dot{x}}{x}\sqrt{2ax+xx}$, *or* $ax^{\frac{1}{2}-1}\dot{x}\sqrt{2a+x}$ *be required.*

This belongs to the 9th and 13th Forms. To get the Fluent of $\frac{x^{\frac{1}{2}-1}\dot{x}}{\sqrt{2a+x}}$ by Form 9th; here $\alpha=2a$, $\beta=1, z=x, \eta=1$: Whence the Fluent $= 2.3025$ Log: $a+x+\sqrt{2ax+xx} := \phi$. The given Fluxion therefore would become $ax^{\pi}\dot{x}\times\overline{2a+x}^{\mu+\tau}$. Where $\pi=-\frac{1}{2}$, $\mu=-\frac{1}{2}$, $\tau=1$, $\alpha=2a$, $\beta=1$, $\eta=1$, $z=x$, $\gamma=2a+x$, $\delta=0$: And the Fluent of $x^{-\frac{1}{2}}\dot{x}\sqrt{2a+x}$ is $=a\phi+x^{\frac{1}{2}}\sqrt{2a+x}$, by Form 13th: And the Fluent of $ax^{-\frac{1}{2}}\dot{x}\sqrt{2a+x}=aa\phi+a\sqrt{2ax+xx}$.

Ex. 6.

To find the Fluent of $\frac{bz^2\dot{z}}{aa-\frac{aa}{bb}z^2}$.

This will be had by Form 6th and 11th. To find the Fluent of $\frac{\dot{z}}{aa-\frac{aa}{bb}z^2}$ by Form 6th. Here $\alpha=aa$, $\beta=\frac{aa}{bb}$, $\eta=2$, then the Fluent is $=\frac{b}{2aa}\times$ $2.3025\ \text{Log.}\frac{a+\frac{a}{b}z}{a-\frac{a}{b}z} = \frac{b}{aa}\times 2.3025\times\text{Log.}\sqrt{\frac{b+z}{b-z}}$ $=\phi$. Then by Form 11th, $\alpha=aa$, $\beta=-\frac{aa}{bb}$, $\mu=-1$, $\pi=0$, $\lambda=1$, $\gamma=1$, $\epsilon=3$, $\delta=3$, whence the Fluent of $\frac{z^2\dot{z}}{aa-\frac{aa}{bb}z^2}$ is $=\frac{-aa\times 1}{-\frac{aa}{bb}\times 1}\phi$

+

$+\frac{bbz}{-aa} = bb\varphi - \frac{bb}{aa}z$: and consequently the Fluent of $\frac{bz^2\dot{z}}{aa - \frac{aa}{bb}z^2}$ is $= b^3\varphi - \frac{b^3}{aa}z = \frac{b^4}{aa} \times 2\cdot3025 \times$ Log: $\sqrt{\frac{b+z}{b-z}}: - \frac{b^3}{aa}z$.

Ex. 7.

To find the Fluent of $b\dot{x}\sqrt{\frac{4a-3x}{a-x}}$.

This by Form 23d is transformed into $b \times \overline{a+3v}^{\frac{1}{2}} \times \frac{v^{-\frac{1}{2}}\dot{v}}{-1}$. Where $\lambda = 1$, $e+fz^\theta = a-x=v$, $g+hz^\theta = 4a-3x$, $p=a$: By Form the 9th the Fluent of $\frac{v^{-\frac{1}{2}}\dot{v}}{\sqrt{a+3v}}$ is $\frac{2}{\sqrt{3}} \times 2.30258 \times$ Log: $\overline{3v+\sqrt{a+3v}}:=\varphi$. And by Form 13th, the Fluent of $-bv^{-\frac{1}{2}}\dot{v}\sqrt{a+3v}$ is $= -\frac{ba}{2}\varphi - bv^{\frac{1}{2}}\sqrt{a+3v} = -b\sqrt{\overline{a-x} \times \overline{4a-3x}}$ $-\frac{2.30258ba}{\sqrt{3}} \times$ Log: $\overline{3a-3x+\sqrt{4a-3x}}$:

Ex. 8.

To find the Fluent of $\frac{x^{-1}\dot{x}\sqrt{aa-xx}}{bb+\frac{b}{a}xx}$.

This by Division is reduced to $\frac{x^{-1}\dot{x}}{bb}\sqrt{aa-xx}$ $-\frac{x\dot{x}}{ba} \times \frac{\sqrt{aa-xx}}{bb+\frac{b}{a}xx}$. The Fluent of $\frac{x^{-1}\dot{x}}{bb}\sqrt{aa-xx}$ (by Form 9th and 13th) is $= \frac{\sqrt{aa-xx}}{bb} - \frac{2.30258a}{bb}$ $\times$ Log.

$\times$ Log. $\dfrac{a+\sqrt{aa-xx}}{x}$. And by Form the 23d, $\dfrac{x\dot{x}\sqrt{aa-xx}}{bb+\frac{b}{a}xx}$ is transform'd to $\overline{p+\frac{b}{f}v}\big|^{-1}\times\dfrac{v^{\frac{1}{2}}\dot{v}}{\theta f}=$ $\dfrac{-\frac{1}{2}v^{\frac{1}{2}}\dot{v}}{bb+ba-\frac{b}{a}v}$; For $g+hz^{\theta}=bb+\frac{b}{a}xx$, $e+fz^{\theta}=aa-xx=v$, $p=bb+ba$, $\theta f=-2$, $m=\frac{1}{2}$, $r=-1$, $\lambda=1$. Now by Form the 6th, the Fluent of $\dfrac{v^{-\frac{1}{2}}\dot{v}}{p-\frac{b}{a}v}$

$$=2.30258\sqrt{\frac{a}{pb}}\times\text{Log. of }\frac{p+\frac{b}{a}v+2\sqrt{\frac{pbv}{a}}}{p-\frac{b}{a}v}=\phi.$$

And by Form 11th the Fluent of $\dfrac{1}{2ba}\times\dfrac{v^{\frac{1}{2}}\dot{v}}{p-\frac{b}{a}v}$ $=\dfrac{p}{2bb}\phi-\dfrac{\sqrt{v}}{bb}$: Whence the Fluent of $\dfrac{x^{-1}\dot{x}\sqrt{aa-xx}}{bb+\frac{b}{a}xx}$ is $=\dfrac{\sqrt{aa-xx}}{bb}-\dfrac{\sqrt{v}}{bb}+\dfrac{2.3025a}{bb}\text{Log:}\dfrac{a-\sqrt{aa-xx}}{x}$

$$+\frac{p}{2bb}\phi=\frac{2a\times2.30258\text{Log:}\frac{a-\sqrt{v}}{x}:+p\phi}{2bb}.$$

Ex. 9.

The Fluent of $bbx^{-2}\dot{x}\times\dfrac{\overline{a+x}|^{\frac{3}{2}}}{\sqrt{ab+ax}}$ *is required.*

Here $\theta=1$, and the Fluxion would become

 $bbx^{-\theta-1}\dot{x}$

$bbx^{-\theta-1}\dot{x} \times \frac{\overline{a+x^\theta}^{\frac{1}{2}}}{\sqrt{ab+ax^\theta}}$, therefore the given Fluxion is reduced by Multiplication to $\frac{bbaa\dot{x}+2bbax\dot{x}+bbx^2\dot{x}}{xx\sqrt{ab+ax}\times\sqrt{a+x}}$ $=\frac{bb\dot{x}}{\sqrt{ab+ax}\sqrt{a+x}}+\frac{2bbax^{-2}\dot{x}}{\sqrt{a+abx^{-1}}\sqrt{1+ax^{-1}}}+\frac{bba^2x^{-3}\dot{x}}{\sqrt{a+abx^{-1}}\sqrt{1+ax^{-1}}}$; each Term whereof belongs to the 23d Form. Let $e+fz^\theta=a+x=v$, $g+hz^\theta=ab+ax$, $\theta=1$, $r=-\frac{1}{2}$, $m=-\frac{1}{2}$, $p=ab-aa$, then $\frac{bb\dot{x}}{\sqrt{ab+ax}\sqrt{a+x}}$ $=\frac{bbx^{-\frac{1}{2}}\dot{v}}{\sqrt{p+av}}$, whose Fluent by Form the 9th will be $\frac{2bb}{\sqrt{a}}\times 2.30258\,\text{Log}:\overline{\sqrt{av}+\sqrt{p+av}}:=\frac{2bb}{\sqrt{a}}\times 2.30258\times$ $\text{Log}:\overline{\sqrt{aa+ax}+\sqrt{ab+ax}}:=2.302585\times\frac{2bb}{\sqrt{a}}\times\text{Log}:$ of $\overline{\sqrt{a+x}+\sqrt{b+x}}:=\frac{2bb}{\sqrt{a}}A$, the Fluent of the first Term.

Again for the 2d and 3d Terms: make $e+fz^\theta=1+ax^{-1}=v$, $g+hz^\theta=a+abx^{-1}$, $\theta=-1$, $p=a-b$, $\lambda=1$ in the second Term, and $\lambda=2$ in the third Term, which therefore will be transform'd into $-\frac{2bbv^{-\frac{1}{2}}\dot{v}}{\sqrt{p+bv}}$ and $\frac{bbv^{-\frac{1}{2}}\dot{v}-bbv^{\frac{1}{2}}\dot{v}}{\sqrt{p+bv}}$ whose Sum is $-\frac{bbv^{-\frac{1}{2}}\dot{v}}{\sqrt{p+bv}}$ $-\frac{bbv^{\frac{1}{2}}\dot{v}}{\sqrt{p+bv}}$. By Form the 9th the Fluent of $\frac{v^{-\frac{1}{2}}\dot{v}}{\sqrt{p+bv}}$ $=\frac{2}{\sqrt{b}}\times 2.3025\,\text{Log}:\overline{\sqrt{bv}+\sqrt{p+bv}}:=2.30258\times$ $\frac{2}{\sqrt{b}}\times\text{Log}:\sqrt{\frac{ba+bx}{x}}+\sqrt{\frac{ba+ax}{x}}:=\frac{2}{\sqrt{b}}B$. And by

by Form the 11th, the Fluent of $\frac{v^{\frac{1}{2}}\dot{v}}{\sqrt{p+bv}}$ is $=\frac{-p}{b\sqrt{b}}B + \frac{v^{\frac{1}{2}}}{b}\sqrt{p+bv}$, multiply the Sum of these Fluents by $-bb$, and you will have $\frac{\overline{-2bb+bp}}{\sqrt{b}}B - bv^{\frac{1}{2}}\sqrt{p+bv} = \frac{ba-3bb}{\sqrt{b}}B - \frac{b}{x}\sqrt{ab+ax}\sqrt{a+x}$, to which add the first Term $\frac{2bb}{\sqrt{a}}A$, and you will have $\frac{2bb}{\sqrt{a}}A + \frac{ba-3bb}{\sqrt{b}}B - \frac{b}{x}\sqrt{ab+ax}\sqrt{a+x}$ the Fluent required.

Ex. 10.

To find the Fluent of $\frac{\dot{z}}{4bz}\sqrt{a^2b^2 - a^3z + 16b^2z^2}$.

The given Fluxion will be reduced to $\frac{\dot{z}}{4bz} \times \frac{a^2b^2 - a^3z + 16b^2z^2}{\sqrt{a^2b^2 - a^3z + 16b^2z^2}} = \frac{a^2bz^{-2}\dot{z}}{4\sqrt{16b^2 - a^3z^{-1} + a^2b^2z^{-2}}} - \frac{a^3\dot{z}}{4b\sqrt{a^2b^2 - a^3z + 16b^2z^2}} + \frac{4bz\dot{z}}{\sqrt{a^2b^2 - a^3z + 16b^2z^2}}$, each of these Terms belongs to the 27th Form.

For the first Term, $v = -\frac{1}{2}a^3 + a^2b^2z^{-1}$, $p = 16bb - \frac{a^4}{4bb}$, $\lambda = 1$, $m = -\frac{1}{2}$, $\theta = -1$, and it will be transform'd into $\frac{-\dot{v}}{4b\sqrt{p+\frac{vv}{g}}} = \frac{-a\dot{v}}{4\sqrt{pg+vv}}$, whose Fluent by Form the 9th will be $-\frac{a}{4} \times 2.30258\,\text{Logar:}\, v + \sqrt{pg+vv}: = \frac{-a}{4} \times 2.30258 \times \text{Log:}\frac{a^2b^2}{z} - \frac{1}{2}a^3 + \frac{ab}{z}\sqrt{a^2b^2 - a^3z + 16b^2z^2}:$

Again for the other two Terms, here $v = -\frac{1}{2}a^3 + 16b^2z$, $p = a^2b^2 - \frac{a^6}{64b^2}$, $m = -\frac{1}{2}$, $\theta = 1$, $\lambda = 1$ in the second Term, and $=2$ in the third: And these two Terms by Form the 27th will be transform'd into $\frac{-a^3\dot{v}}{16bb\sqrt{pg+vv}}$ and $\frac{v\dot{v}}{16bb\sqrt{pg+vv}} + \frac{a^3\dot{v}}{32bb\sqrt{pg+vv}}$, whose Sum is $\frac{v\dot{v}}{16bb\sqrt{pg+vv}} - \frac{a^3\dot{v}}{32bb\sqrt{pg+vv}}$; whose Fluent (by Form the 9th) $= \frac{\sqrt{pg+vv}}{16bb} - \frac{a^3L}{32bb}$ $\times$ Log. of: $v+\sqrt{pg+vv}$: $= \frac{\sqrt{a^2b^2 - a^3z + 16bbz^2}}{4b}$ $- 2.302585 \times \frac{a^3}{32bb} \times$ Log: $-\frac{1}{2}a^3 + 16bbz + 4b \times \sqrt{a^2b^2 - a^3z + 16b^2z^2}$: Whence the Fluent of $\frac{\dot{z}}{4bz}$ $\sqrt{a^2b^2 - a^3z + 16b^2z^2}$ is $= \frac{1}{4b}\sqrt{a^2b^2 - a^3z + 16bbzz}$ $- \frac{a}{4} \times 2.302585$Log: $\frac{a^2b^2}{z} - \frac{1}{2}a^3 + \frac{ab}{z} \times$ $\sqrt{a^2b^2 - a^3z + 16b^2z^2}$: $- \frac{a^3}{32bb} \times 2.3025$Log: $16bbz$ $-\frac{1}{2}a^3 + 4b\sqrt{a^2b^2 - a^3z + 16b^2z^2}$:

I shall add a few more Examples, chiefly to illustrate the METHOD of REDUCTION.

EX. II.

Let $\frac{dz^{\frac{3}{5}-1}\dot{z}}{e+fz}$ *be proposed*: Divide by z and it will be reduced to $\frac{dz^{-\frac{2}{5}-1}\dot{z}}{f+ez^{-1}}$. This belongs to Form 7th, where $\eta = -1$, $\lambda = 5$, $\delta = 2$, $x = \sqrt[5]{\frac{e}{fz}}$, $K = 36°$; $a = ,587785$; $b = ,951056$; $c = 0$, and $s = ,809017$; $t = -,309017$; $u = -1$: where always observe that the

the Cosines of Arches above 90° are negative Numbers: then will be found P, Q, R, in Degrees; and the Logarithms of $\sqrt{1-2sx+xx}$, $\sqrt{1-2tx+xx}$, $\sqrt{1-2ux+xx}$, for which Logarithms put S, T, V; Then (by Case the 6th) the Fluent is $\frac{d}{-f^{\frac{3}{5}}e^{\frac{2}{5}}}$ × into $-,309017\times 2LS+,951056\times 2NP+,809017\times 2LT-.587785\times 2NQ-LV$.

Ex. 12.

Given $x^{-3}\dot{x}\times\overline{aa-x^2}|^{\frac{2}{3}}$, this by Form 21st (puting $v=aa-x^2$) is transformed into $\overline{v-aa}|^{-2}\times\frac{v^{\frac{2}{3}}\dot{v}}{-2}=-\frac{1}{2}\times\frac{v^{\frac{2}{3}}\dot{v}}{\overline{-aa+v}|^2}$; which belongs to Form 7th and 14th.

Ex. 13.

Let $\frac{x^{-1}\dot{x}\times\overline{aa+xx}|^{\frac{3}{2}}}{\overline{abb-bxx}|^{\frac{3}{2}}}$ *be given*; this is reduced to $\frac{aax^{-1}\dot{x}\times\overline{aa+xx}^{\frac{1}{2}}}{\overline{abb-bxx}|^{\frac{3}{2}}}+\frac{x\dot{x}\times\overline{aa+xx}^{\frac{1}{2}}}{\overline{abb-bx^2}|^{\frac{3}{2}}}$, both which Terms belong to Form 24.

Ex. 14.

Let $\frac{x^2\dot{x}\times\overline{a+x}^{\frac{1}{3}}}{b+x\times\overline{ab-bx}|^{\frac{4}{3}}}$ *be proposed*; divide by $x+b$, and it is reduced to $\frac{x\dot{x}\times\overline{a+x}|^{\frac{1}{3}}}{\overline{ab-bx}|^{\frac{4}{3}}}-\frac{b\dot{x}\times\overline{a+x}^{\frac{1}{3}}}{\overline{ab-bx}|^{\frac{4}{3}}}+\frac{bb\dot{x}\times\overline{a+x}^{\frac{1}{3}}}{b+x\times\overline{ab-bx}|^{\frac{4}{3}}}$: The two first Terms belong to Form 24th, and the last to Form 25th.

Ex. 15.

Let $\frac{y^{-2}\dot{y}}{a+y\sqrt{ab+cy-yy}}$ *be given*; divide by $a+y$, and it will be reduced to $\frac{y^{-2}\dot{y}}{a\sqrt{ab+cy-yy}}$ $-\frac{y^{-1}\dot{y}}{aa\sqrt{ab+cy-yy}}+\frac{\dot{y}}{aa\times\overline{a+y}\sqrt{ab+cy-yy}}$; the laſt Term belongs to the 32d Form, and (by reducing the Index) the two firſt Terms become $\frac{y^{-3}\dot{y}}{a\sqrt{-1+cy^{-1}+aby^{-2}}}-\frac{y^{-2}\dot{y}}{aa\sqrt{-1+cy^{-1}+aby^{-2}}}$, both which belong to the 27th Form.

Ex. 16.

Given $\frac{d\dot{x}\sqrt{aa-bx+xx}}{a-x}$; this (by Multiplication) is reduced to $\frac{daa\dot{x}}{\overline{a-x}\sqrt{aa-bx+xx}}+\frac{dx^2\dot{x}-bdx\dot{x}}{\overline{a-x}\sqrt{aa-bx+xx}}$. The firſt Term belongs to Form 32d. And dividing the reſt by $-x+a$, they are reduced to $\frac{-dx\dot{x}}{\sqrt{aa-bx+xx}}$ $+\frac{\overline{bd-ad}\times\dot{x}}{\sqrt{aa-bx+xx}}+\frac{\overline{aad-abd}\times\dot{x}}{\overline{a-x}\sqrt{aa-bx+xx}}$. The laſt belongs to Form 32, and the two former are Fluxions of the 27th Form.

Ex. 17.

Given $\frac{x^{-\frac{1}{3}}\dot{x}\times\overline{ab-xx}|^{\frac{1}{3}}}{a^4-3aax^2+x^4}$; this (by reducing the Index) becomes $\frac{x^{-5}\dot{x}\times\overline{-1+abx^{-2}}|^{\frac{1}{3}}}{1-3aax^{-2}+a^4x^{-4}}$, which belongs to Form 33d.

Ex. 18.

Let $\frac{z^{-1}\dot{z}\sqrt{aa-zz}}{a^4-aaz^2+z^4}$ *be proposed*; because $4a^8$ is greater than a^4, it cannot be resolved by the 33d Form, therefore I divide by $a^4-aazz+z^4$, which reduces it to $\frac{z^{-1}\dot{z}\sqrt{aa-zz}}{a^4}+\frac{z\dot{z}\sqrt{aa-zz}}{aa\times\overline{a^4-a^2z^2+z^4}}$ $-\frac{z^3\dot{z}\sqrt{aa-zz}}{a^4\times\overline{a^4-a^2z^2+z^4}}$; the two last Terms belong to Form the 34th; and the first is further reduced to $\frac{z^{-2}\dot{z}}{aa\sqrt{1-aaz^{-2}}}-\frac{z\dot{z}}{a^4\sqrt{a^2-z^2}}$, which belong to Forms 10th and 3d.

Ex. 19.

Let there be given $\frac{x\dot{x}}{\overline{aa-ax+xx}\sqrt{aa-ax}}$; this is reduced to $\frac{aax\dot{x}}{\overline{aa-ax+xx}\times\overline{aa-ax}|^{\frac{3}{2}}}-\frac{ax^2\dot{x}}{\overline{aa-ax+xx}\times\overline{aa-ax}|^{\frac{3}{2}}}$; the former Term belongs to Form the 35th; and the latter by Division is further reduced to $\frac{-a\dot{x}}{\overline{aa-ax}|^{\frac{1}{2}}}$ $+\frac{a^3\dot{x}-a^2x\dot{x}}{\overline{aa-ax+xx}\times\overline{aa-ax}|^{\frac{3}{2}}}$; the first belongs to the binomial Forms, and the rest to Form 35; where observe that the Part first found (belonging to Form 35) destoys a Part of this last Fluxion; and then the whole is reduced to this, $\frac{a^3\dot{x}}{\overline{aa-ax+xx}\times\overline{aa-ax}|^{\frac{3}{2}}}$ $-\frac{a\dot{x}}{\overline{aa-ax}|^{\frac{1}{2}}}$.

Ex. 20.

Given $\frac{z^{rn-1}\dot{z}}{a+bz^n+cz^{2n}+dz^{3n}}$, extract one Root h of the Denominator (supposed to be put $=0$), then will $h-z^n=0$, by this divide the said Denominator, and

and let the Quotient be $e+fz^n+gz^{2n}$; then the given Fluxion is reduced to this $\dfrac{z^{rn-1}\dot{z}}{\overline{b-z^n}\times\overline{e+fz^n+gz^{2n}}}$, which belongs to the 29th, 30th, or 31st Form.

Ex. 21.

There is given $\dfrac{z^{rn-1}\dot{z}}{a+bz^n+cz^{2n}+dz^{3n}+tz^{4n}}$, extract two Roots a, b of the Denominator; then you have $a-z^n=0$, and $b-z^n=0$, divide the Denominator by $\overline{a-z^n}\times\overline{b-z^n}$, and let the Quotient be $e+fz^n+gz^{2n}$. Then the given Fluxion becomes $\dfrac{z^{rn-1}\dot{z}}{\overline{a-z^n}\,.\,\overline{b-z^n}\,.\,\overline{e+fz^n+gz^{2n}}}$ $= \dfrac{z^{rn-1}\dot{z}}{s\times\overline{b-z^n}\times\overline{e+fz^n+gz^{2n}}} - \dfrac{z^{rn-1}\dot{z}}{s\times\overline{a-z^n}\times\overline{e+fz^n+gz^{2n}}}$ (putting $a-b=s$,) both which Terms belong to the 29th, 30th, or 31st Form.

Ex. 22.

Given $\dfrac{z^{rn-1}\dot{z}}{\overline{a+bz^n+kz^{2n}}\times\overline{e+fz^n+gz^{2n}}|^m}$, here r and $2m$ must be Integers; put $p=\sqrt{bb-4ak}$, and the Fluxion is resolved into these two Terms, $\dfrac{k}{p}\times$ $\dfrac{z^{rn-1}\dot{z}}{\overline{\frac{b-p}{2}+kz^n}\,.\,\overline{e+fz^n+gz^{2n}}|^m} - \dfrac{k}{p}\times\dfrac{z^{rn-1}\dot{z}}{\overline{\frac{b+p}{2}+kz^n}\,.\,\overline{e+fz^n+gz^{2n}}|^m}$ then dividing by the Denominators $\dfrac{b-p}{2}+kz^n$, and $\dfrac{b+p}{2}+kz^n$, if there be Occasion; and the resulting Terms will belong to the 32d Form, and perhaps to some others of the foregoing Forms.

SCHOLIUM.

SCHOLIUM.

Although the Construction of many of the foregoing Forms depends on several Things that have not yet been deliver'd; yet it may be proper in this Place to give a short Account thereof, that the Reader may not be intirely ignorant of them, which he will the better understand after he has read such Parts of the following Book as they are founded on.

1. It is plain by Prop. III. that the Fluxions in Form 1, 3, 19 and 20 belong to the Fluents there assigned.

2. Form 16th is calculated in Example 13 Prop. X. and Form the 18th in Example 35. Likewise the Fluent in Form the 15th is found by Prop. X. Assuming $Az^{\pi+1} + Bz^{\pi+\eta+1} + Cz^{\pi+2\eta+1}$ &c $\times$ $\overline{\alpha+\beta z^{\eta}}^{\mu+1}$ = Fluent of $\overline{\alpha+\beta z^{\eta}}^{\mu} z^{\pi}\dot{z}$.

3. By Prob. II. Sect. II. the Fluxion of any Quantity divided by that Quantity is the Fluxion of the Hyperbolic Logarithm of that Quantity; and that the Number 2.302585 reduces the common to the hyperbolic Logarithms. Now if the Fluents in Form 2, 4, 6 and 9th be put into Fluxions according to the aforesaid Rule, it will appear that they will respectively produce the Fluxions in these Forms.

4. In Example 3d, Prob. X. Sect. II. it is shewn, that the hyperbolic Space between the Assymptotes is = Fluent of $\frac{ab}{a+x}\dot{x}$, in which if you write RR for ab, Rz (or z) for $a+x$, and R$\dot{z}$ (or $\dot{z}$) for $\dot{x}$, you will have $RR\frac{\dot{z}}{z}$, whose Fluent is the hyperbolic Space in Form the 2d. Likewise if you write RR for ab, $\frac{\alpha}{\beta}+z^{\eta}$ for $a+x$, and $\eta z^{\eta-1}\dot{z}$ for $\dot{x}$, you will have $\frac{\beta\eta RRz^{\eta-1}\dot{z}}{\alpha+\beta z^{\eta}}$ as in Form the 4th. And lastly, write

RR for ab, and $\sqrt{\alpha}+\sqrt{\beta}z^{n}$ for $a+x$, and $\sqrt{\beta}\times\frac{1}{2}nz^{\frac{1}{2}n-1}\dot{z}$ for $\dot{x}$; and likewise write $\sqrt{\alpha}-\sqrt{\beta}z^{n}$ for $a+x$, and $-\sqrt{\beta}\times\frac{1}{2}nz^{\frac{1}{2}n-1}\dot{z}$ for $\dot{x}$, and the Difference of these Fluxions will be $\dfrac{nRR\sqrt{\alpha\beta}\times z^{\frac{1}{2}n-1}\dot{z}}{\alpha-\beta z^{n}}$ as in Form the 6th.

5. By Ex. 9. Prob. X, Sect. II. the hyperbolic Sector is equal to the Fluent of $\dfrac{bba}{2}\times\dfrac{\dot{t}}{bb-tt}$, in which writing R for a and b, $R\sqrt{\dfrac{\beta z^{n}}{\alpha}}$ for t and $\dfrac{n}{2}R\sqrt{\dfrac{\beta}{\alpha}}\times z^{\frac{n}{2}-1}\dot{z}$ for $\dot{t}$; you'll have $\dfrac{nRR\sqrt{\alpha\beta}\times z^{\frac{1}{2}n-1}\dot{z}}{4\times\overline{\alpha-\beta z^{n}}}$ as in the 6th Form.

6. And by the same Example, the hyperbolic Sector is $=$ Fluent of $\dfrac{ab\dot{y}}{2\sqrt{ab+yy}}=$ (writing R for a and b) $\dfrac{RR\dot{y}}{2\sqrt{RR+yy}}$; now if you write $R\sqrt{\dfrac{\beta z^{n}}{\alpha}}$ for y, and $\frac{1}{2}nR\sqrt{\dfrac{\beta}{\alpha}}\times z^{\frac{1}{2}n-1}\dot{z}$ for $\dot{y}$, you will have $\dfrac{nRR\sqrt{\beta}\times z^{\frac{1}{2}n-1}\dot{z}}{4\sqrt{\alpha+\beta z_{n}}}$, in like manner if you put $\dfrac{2R}{\alpha}\sqrt{\alpha\beta z^{n}+\beta^{2}z^{2n}}=y$, or $\dfrac{2R}{-\alpha}\sqrt{\alpha\beta z^{n}+\beta^{2}z^{2n}}=y$, when α is a negative Quantity, or $\dfrac{R}{\sqrt{-\alpha}}\sqrt{\alpha+\beta z^{n}}=y$, when α is negative, and expunge y and $\dot{y}$, you will get the Fluxion of the 9th Form any of these Ways.

7. Since the Length of any Arch of a Circle (whose Radius is R) is $=\dfrac{3.1416R}{180}\times$ Degrees in that Arch; or Length of the Arch $=,017453R\times$ Degrees in that Arch: And by Example the 5th, Prob. VIII. Sect. II. The Arch $=$ Fluent of $\dfrac{RR\dot{t}}{RR+tt}$; if you write $R\sqrt{\dfrac{\beta z^{n}}{\alpha}}$ for

for t, and $\frac{\eta R}{2}\sqrt{\frac{\beta}{\alpha}} \times z^{\frac{1}{2}\eta-1}\dot{z}$ for $\dot{t}$, you will have $\frac{\eta R\sqrt{\alpha\beta} \times z^{\frac{1}{2}\eta-1}\dot{z}}{2 \times \overline{\alpha+\beta z^{\eta}}}$, whose Fluent therefore is $=$ that Arch $=$,017453R $\times$ Degrees in the Arch $= \frac{2\ \textit{Sectors}}{R}$, from all which the Construction of Form the 5th will easily appear.

8. In like Manner by the same Problem, the Arch $=$ Fluent of $\frac{R\dot{y}}{\sqrt{RR-yy}}$, in which writing for y, $R\sqrt{\frac{\beta z^{\eta}}{\alpha}}$, or $\frac{2R}{\alpha}\sqrt{\alpha\beta z^{\eta}-\beta^2 z^{2\eta}}$, and the Fluxion for $\dot{y}$, you will get $\frac{\eta R\sqrt{\beta}}{2} \times \frac{z^{\frac{1}{2}\eta-1}\dot{z}}{\sqrt{\alpha-\beta z^{\eta}}}$, and $\eta R\sqrt{\beta} \times \frac{z^{\frac{1}{2}\eta-1}\dot{z}}{\sqrt{\alpha-\beta z}}$, from which all the other Varieties in Form the 10th are easily deduced.

9. Form the 11th is gradually calculated by Cor. 2. Prop. IV. where it is demonstrated that the Fluent of $\overline{\alpha+\beta z^{\eta}}^{\mu} z^{p+\eta}\dot{z}$ is $= \frac{z^{p+1} \times \overline{\alpha+\beta z^{\eta}}^{\mu+1} - \overline{p+1}.\alpha A}{\overline{\mu\eta+\eta+p+1} : \times \beta}$, in which writing π for p, φ for A, γ for $\overline{\alpha+\beta z^{\eta}}^{\mu+1}$, you will have the Fluent of $\overline{\alpha+\beta z^{\eta}}^{\mu} z^{\pi+\eta}\dot{z}$; again, writing $\pi+\eta$ for p, Fluent of $\overline{\alpha+\beta z^{\eta}}^{\mu} z^{\pi+\eta}\dot{z}$ (now found) for A; you'll have the Fluent of $\overline{\alpha+\beta z^{\eta}}^{\mu} z^{\pi+2\eta}\dot{z}$; again, writing $\pi+2\eta$ for p, and the Fluent of $\overline{\alpha+\beta z^{\eta}}^{\mu} z^{\pi+2\eta}\dot{z}$ (last found) for A, you get the Fluent of $\overline{\alpha+\beta z^{\eta}}^{\mu} z^{\pi+3\eta}\dot{z}$, and so on.

In like Manner the 12th Form is calculated from Cor. 3. Prop. IV. where the Fluent of $\overline{\alpha+\beta z^{\eta}}^{\mu} z^{p}\dot{z} = \frac{z^{p+1}\gamma - \overline{\mu\eta+\eta+p+1} \times \beta B}{\overline{p+1} \times \alpha}$. And by a like Process the

the 13th and 14th Forms are gradually calculated from Cor. 4 and 5, Prop. IV. Theſe 4 Forms might have been differently expreſs'd from what they are in the Table. For Example, Form the 11th may be expreſſed thus; let $e = \pi + 1$, $d = \pi + 1 + n + \mu n$, then the Fluent of $\overline{\alpha + \beta z^n}|^{\mu} z^{\pi + \lambda n} \dot{z} =$

$$\frac{\overline{-\alpha)}^{\lambda} \times e \times \overline{e+n} \times \overline{e+2n} \times \overline{e+3n}.\ \&c.\ \text{to } \lambda \text{ Places}}{\beta^{\lambda} \times d \times \overline{d+n} \times \overline{d+2n} \times \overline{d+3n}.\ \&c.\ \text{to } \lambda \text{ Places}} \times \varphi$$

$$-\frac{z^e \gamma}{e \alpha \varphi} A - \frac{d \beta z^n}{\overline{e+n}.\alpha} B - \frac{\overline{d+n}.\beta z^n}{\overline{e+2n}.\alpha} C - \frac{\overline{d+2n}.\beta z^n}{\overline{e+3n}.\alpha} D - \&c$$

continued to $\lambda + 1$ Places; where A, B, C, &c, are the firſt, ſecond, third, &c Terms with their Signs, $\varphi =$ Fluent of $\overline{\alpha+\beta z^n}|^{\mu} z^{\pi} \dot{z}$, $\gamma = \overline{\alpha+\beta z^n}|^{\mu+1}$.

10. The 17th Form is derived from the 11th; for when z and γ are 0, all the Terms vaniſh except the firſt, in which taking 0, 1, 2, 3, &c, Terms (for each of the Terms in the Quantity $e + fz^n + gz^{2n}$ &c) and multiplying each by its reſpective Coefficient, and the Sum of all by φ, you will at laſt obtain this Form.

11. The Fluxions in the 21ſt and following Forms are transform'd by Prop. IX. and their Truth will appear by the bare Subſtitution of the Quantities therein contain'd.

12. There ſtill remain the 7th and 8th Forms, whoſe Calculus is ſomething more difficult: Theſe I inveſtigate after the following manner. Let T, U, W, &c, ſtand for the Quantities $\sqrt{1-2sx+xx}$, $\sqrt{1-2tx+xx}$, $\sqrt{1-2ux+xx}$, &c; then the Fluxion of L × Log. T (or the hyperbolic Log. T) will be $\frac{-s\dot{x}+x\dot{x}}{1-2sx+xx}$, and the like for U, W, &c; as is plain from what is delivered in Art. 3. Again, the Fluxion of NP (as will appear by Art. 8) is $\frac{a\dot{x}}{1-2sx+xx}$, and ſo for the reſt NQ, NR, &c. Laſtly, we muſt take for granted (for I ſhall not ſtay to demonſtrate it here)

here) that the Product of all the Squares of T, U, W, &c, if λ is an even Number, or the last drawn into the Squares of all the rest, if λ is an odd Number, is always equal to $1+x^\lambda$. These Things premised, let us investigate any one Case; as for Example, to find the Fluent of $\frac{\dot{x}}{1+x^3}$; here $\lambda=3$ and $\frac{180}{3}=60$: here b, c, u, &c, $=0$, and $t=-1$, and therefore $\sqrt{1-2tx+xx}$ $=1+x$: and all Arches above 180 are excluded; hence I assume $LA \times \text{Log.}\sqrt{1-2sx+xx} + BNP + CL \times \text{Log.}\,\overline{1+x} =$ Fluent of $\frac{\dot{x}}{1+x^3}$. This in Fluxions gives

$$\frac{Ba-As+Ax}{1-2sx+xx}\dot{x}+\frac{C\dot{x}}{1+x}=\frac{\dot{x}}{1+x^3};\text{ put } Ba-As=I.$$

then $\frac{I+Ax}{1-2sx+xx}+\frac{C}{1+x}=\frac{1}{1+x^3}$; reduce them to a common Denominator, then

$$\frac{\begin{array}{ccc} +I & +Ax & +Ax^2 \\ +C & \begin{array}{c}+I\\-2sC\end{array} & +C \end{array}}{\begin{array}{cccc} 1 & -2sx & +1\,x^2 & +x^3 \\ & +1 & -2s & \end{array}}=\frac{1}{1+x^3}.$$

Here the Denominators being equal, we have $1-2s$ $=0$, or $2s=1$; and equating the Coefficients of the homologous Terms in the Numerator, $I+C=1$. $A+I$ $-2sC$ or $A+I-C=0$, and $A+C=0$. From whence will be found $A=-\frac{1}{3}$, $C=\frac{1}{3}$, $I=\frac{2}{3}$, $B=\frac{I}{2a}=$ (because $aa=\frac{3}{4}$,) $\frac{2}{3}a$. Whence the Fluent is $-\frac{1}{3}L \times$ $\text{Log.}\sqrt{1-2sx+xx}+\frac{2}{3}aNP+\frac{1}{3}L \times \text{Log.}\,\overline{1+x}$.

This done, I put $x^\lambda=\frac{\beta z^n}{\alpha}$; and (by Prop. IX.) transform the Fluxion $\frac{\dot{x}}{1+x^3}$ into this other $\frac{n\alpha^{\frac{2}{3}}\beta^{\frac{1}{3}}}{3} \times$ $\frac{z^{\frac{1}{3}n-1}\dot{z}}{\alpha+\beta z^n}$, whose Fluent therefore you have above.

In

In equating the Coefficients, if you had made $A+I-C=1$, and the other Equations $=0$, you would have got the Fluent of $\frac{x\dot{x}}{1+x^3}$.

After the ſame manner may the 8th Form be inveſtigated, remembring what was ſaid of Sines and Logarithms, and obſerving that (if you put T, U, W, &c. for $1-x$, $\sqrt{1-2sx+xx}$, $\sqrt{1-2tx+xx}$) the firſt T into the Squares of all the reſt, if λ is an odd Number; or the firſt and laſt into the Squares of the reſt, if λ is an even Number, is always equal to $1-x^\lambda$: The Demonſtration of which belongs not to this Place.

Here note that in theſe two Forms, you may if you pleaſe put P, Q, &c for the Degrees of Arches whoſe Sines are $\frac{a}{\sqrt{1-2sx+xx}}$, $\frac{a}{\sqrt{1-2tx+xx}}$ &c, their Fluxions being the ſame as the other. But then if the Arch whoſe Sine is $\frac{ax}{\sqrt{1-2sx+xx}}$ be leſs than a Quadrant, the Arch whoſe Sine is $\frac{a}{\sqrt{1-2sx+xx}}$ will be greater than a Quadrant.

PROP. XII.

To correct the Fluent of a given fluxionary Equation or Proportion.

When the Fluent is obtained, by either of the foregoing Propoſitions, from any given fluxionary Equation or Proportion; it is only obtain'd in general. But ſince the Deſign in any particular Problem is to find the contemporary Fluents; ſuch general Equation or Proportion

Proportion therefore is for the most part imperfect till it be corrected by the following

Rule.

1. Instead of each several variable Quantity (or Member) in the Fluent, *substitute* such a *determinate Value* thereof, as each of them is known to have in any one certain and *particular Point* or Place: Then ~~*substitute*~~ Subtract each *Side* of the resulting Equation from the *corresponding Side* of the *Fluent*; and the remaining Equation will be the *correct Fluent*. And the same Rule obtains when the Fluent is expressed by a general Proportion.

2. Or any particular Part of the Fluent may be otherwise had thus without the foregoing Correction: *Substitute* the Values of the variable Quantities for any particular *Time*, Place or Point; do the same for another given Time or Place; and the *Difference* of the resulting Equations, gives the corresponding Part of the *Fluent*.

3. Sometimes it may be sufficient to add some *given Quantity* on one Side of the Equation, which may afterwards be determin'd according to the Nature and Circumstances of the Question.

Demonstration.

Let $X = Z$ be the Fluent obtained in general, from a given fluxionary Equation, $\dot{X} = \dot{Z}$. Now since X may not be equal to Z (by Cor. 2. Prop. II.), take d a given Quantity, and let $X = Z + d$ be an Equation for the contemporary Fluents. Now at a certain Time when $X = b$, let $Z = c$; then you will have the particular Equation $b = c + d$ at that Time; this Equation taken from the former will leave this general Equation for the contemporary Fluents, $X - b = Z + d - c - d$, that is $X - b = Z - c$. Q. E. D.

Scholium.

These Things are to be thus understood when the variable Quantity in the Fluent continually increases (or

(or decreases). But in Case it increases and decreases by Turns, or passes through one or more *Maximums* or *Minimums*; then the several Parts of the Fluent, between any given Point and each Maximum or Minimum must be *separately* found by distinct Operations, and each *corrected* by this Proposition, and then the several Parts *collected* into one *Sum*.

Example 1.

Let $a\dot{x}=2y\dot{y}$, and the Fluent is $ax=\dot{y}y$, now when $y=0$, let $x=0$, and then $ax-0=y^2=0$, or $ax=yy$, which therefore needs no Correction.

Ex. 2.

Again let $a\dot{x}=2y\dot{y}$, and finding the Fluent $ax=yy$. Now when $y=0$, let $x=b$; and the Equation becomes $ab=0$, this substracted from the other Equation, leaves $ax-ab=yy$, the contemporary Fluent required.

Ex. 3.

Let $b\dot{x}-x\dot{x}=y\dot{y}$, the Fluent is $bx-\frac{x^2}{2}=\frac{y^2}{2}$, but at a certain Point of Time $x=b$, and also $y=r$, then the Equation becomes $bb-\frac{1}{2}bb$ or $\frac{1}{2}bb=\frac{rr}{2}$, therefore the correct Fluent is $bx-\frac{1}{2}xx-\frac{1}{2}bb=\frac{1}{2}yy-\frac{1}{2}rr$. That is by Reduction $rr-bb+2bx-xx=yy$.

Ex. 4.

Suppose $2by\dot{y}=\frac{3cx^2\dot{x}}{z}-\frac{cx^3\dot{z}}{z^2}$, the Fluent is $byy=\frac{cx^3}{z}$; but when $y=r$, the Quantity $\frac{cx^3}{z}=A$. Therefore the correct Fluent is $b\times\overline{y^2-r^2}=\frac{cx^3}{z}-A$.

Ex. 5.

Let $\frac{\dot{y}^2}{2b}=\frac{\dot{x}\ddot{x}}{\sqrt{\dot{y}^2+\dot{x}^2}}$, *supposing* $\dot{y}$ *given*; the

Fluent

Fluent is $\frac{y\dot{y}}{2b} = \sqrt{\dot{x}^2 + \dot{y}^2}$; but when $y = a$, $\dot{x} = 0$, and $\dot{y} = \sqrt{\dot{y}^2 + \dot{x}^2}$, therefore the contemporary Fluent is $\frac{y\dot{y} - a\dot{y}}{2b} = \sqrt{\dot{x}^2 + \dot{y}^2} - \dot{y}$, or $\overline{y - a + 2b} \times \dot{y} = 2b\sqrt{\dot{x}^2 + \dot{y}^2}$.

Ex. 6.

Let $b\dot{y} = \frac{\dot{x}}{\sqrt{aa + xx}}$; By Form the 9th its Fluent is $by = 2.\ 3025$ Log. $x + \sqrt{aa + xx}$: but when $y = a$, $x = 0$, then the Equation is $ba = 2.\ 3025$ Log. a. Therefore the corrected Fluent is $by - ba = 2.\ 3025 \times$ Log. $\overline{x + \sqrt{aa + xx}} - 2.\ 3025$ Log. a, that is $by - ba = 2.\ 302585$ Log. $\frac{x + \sqrt{aa + xx}}{a}$.

PROP. XIII.

To investigate a Problem by the Method of Fluxions.

RULES.

1. Let all the Quantities be denoted by proper Symbols, as is explained in the Notation of Fluxions, and let some *one* of the variable Quantities (with which the others may always be compared) be supposed to increase uniformly: And this may be called the *Principal* variable Quantity. Then the given Equations, or such Equations as are deduced from the Conditions of the Problem, must be turn'd into Fluxions, second Fluxions, &c. by Prop. III. in order to get as many Equations of these Fluxions, as you have Occasion for.

2. But becauſe ſometimes ſome Doubt may ariſe about the Signs of the Fluxions: Obſerve that any fluxionary Equations, deduced from the Equations of Curves, or from any given Equations in the Problem, will contain the Fluxions with their proper Signs. But in any Proportions made between Fluxions or Moments, as in ſimilar fluxionary Triangles and the like; then the Fluxions or Moments of Quantities that decreaſe muſt be actually made *negative*; and thoſe that increaſe muſt be written *affirmative*: Or, which is the ſame thing, (ſince one Part of any Whole increaſes whilſt the other Part of it decreaſes, therefore) inſtead of the negative Fluxion, you may take the proper Fluxion of the increaſing Part of that Quantity.

3. Since *Velocity* is always meaſured by the *Spaces* uniformly deſcrib'd thereby; ſo may the *Fluxions* be meaſured by the *Moments* uniformly generated by the Fluxions. Therefore the Moments (uniformly generated or, which amounts to the ſame thing, conſider'd as *ariſing* or *vaniſhing*) may be put for the Fluxions, and the Reſult will always be the very ſame in all Operations. And ſince in many Caſes the reaſoning and calculating with the *Moments* will be more eaſy and evident than with the *Fluxions*, the Equations gained thereby muſt at laſt be changed into fluxionary Equations, by ſubſtituting the Fluxions inſtead of the Moments, which muſt always be ſuppoſed to be taken in the *firſt Inſtant* of their Generation: Or, *at leaſt* when the Operation is over, theſe *Moments* muſt be ſuppoſed to be *diminiſh'd ad infinitum* that their *firſt Ratio* may be always obtain'd.

4. In the Reſolution of any *Problem*, the Nature and Conditions of it are to be cloſely examin'd and ſtrictly purſued according to all the known Methods of *Algebraic Reaſoning*, by attentively conſidering the *Relations* of the Quantities, and their mutual *Proportions* and *Dependance* on one another; and forming your Proceſs according to theſe their Properties, by duly comparing together the *Quantities*, their *Mo-*

ments or *Increments*, their *Fluxions* or *ſecond Fluxions* &c, as the Caſe requires; till you get a competent Number of *Equations*, or *general Proportions*. And then you muſt proceed to expunge ſuch Quantities as are ſuperfluous, till at laſt you get a fluxionary *Equation* or *Proportion* with the Quantities required. Then if there be Occaſion,

5. Find the *Fluent* of the ſaid Equation, or general Proportion, by Prop. X. or XI. and correct it by Prop. XII. And then you have a *compleat Equation*, or *general Proportion*, containing the Quantities ſought.

6. But to obtain an *Equation* of the indetermined Quantities, by having the correct general *Proportion* before found, or by having only the fluxionary Proportion; you muſt aſſign to each indetermined Quantity in the ſaid Proportion (or in the Fluxion thereof) ſuch a determined Value as it is known to have in any particular Caſe; and from thence you muſt draw an Analogy from the *Fluxion* alone (of the general Proportion) or from the *correct Fluent* alone, (or ſometimes from both together) from whence there will be had an *Equation* between the Quantities required; or at leaſt between their Fluxions, whoſe *Fluent* muſt then be *found* and *corrected*. And note, theſe *determined Values* (of the Quantities) may be either expreſſed in Numbers or Symbols, as any one ſhall think proper.

Sometimes it may be ſufficient to aſſume a *given Quantity*, by which multiplying one Side of the Proportion, it will be turned into an *Equation*; and this given Quantity may afterwards be determined according to the Nature of the Queſtion.

Theſe are the *general Rules*, but after all, many things muſt be left to the Sagacity and Invention of the Artiſt.

Corol. Hence every *Problem* belongs to *Fluxions*, in which the *Increments*, or the *Proportions* of the Increments or Moments of the ſeveral variable Quantities contained therein, can in *all Caſes* be computed and expreſſed by *Equations*.

Example 1.

To find the Velocity wherewith the Ordinate BM of a Circle increases in every Point, whilst it moves uniformly along the Diameter AD.

FIG. 1. Let $AD=2r$, $AB=x$, $BM=y$. and by the Nature of the Circle $2rx-xx=yy$; this Equation put into Fluxions gives $2r\dot{x}-2x\dot{x}=2y\dot{y}$, or $\frac{r-x}{y}\dot{x}=\dot{y}$, a general Equation for the Increase of y in all Points. Therefore in A where y is o, and x is o, $\frac{r-x}{y}\dot{x}=\dot{y}$ becomes $\frac{r\dot{x}}{0}=\dot{y}$, therefore $\dot{y}$ is infinite. If $CB=BM$, or $r-x=y$, then $\dot{x}=\dot{y}$, and x and y increase equally. But in C where $r-x=0$, then $\dot{y}=0$, therefore y does not increase at all. In b where $Cb=bm$, then $\dot{y}=-\dot{x}$, therefore y decreases as fast as x increases. Lastly in D where $y=0$, and $x=2r$, $\dot{y}=\frac{-r\dot{x}}{0}=-$Infinity, and there the Ordinate decreases infinitely: And in all the intermediate Points it has all the intermediate Degrees of Increase or Decrease.

Ex. 2.

To find the Space a descending Body will describe in any Time by the uniform Force of Gravity.

Let $z=$ Space, $x=$ Time, $v=$ Velocity, acquired in that Time. By the Principles of Mechanics $\dot{z} \propto v\dot{x} \propto x\dot{x}$, and therefore $\dot{z} \propto v\dot{x} \propto x\dot{x}$; that is the Fluxion of the Space is every where as the Velocity into the Fluxion of Time; that is, (because the Velocity is as the Time) as the Time into the Fluxion of the Time.

Now if only the Ratio and Space be required, it will be sufficient to take the Fluent; and then $z \propto \frac{x^2}{2}$, or

or $z \propto x^2$, that is, the Space is always as the Square of the Time.

But if the abſolute Quantity of the Space is ſought, we muſt reduce the general Proportion or its Fluxion to an Analogy from ſome particular known Caſe. Thus it is known by Experiment, that in the Time t or 1 Second, a Body would acquire ſuch a Velocity as to move through s or $32\frac{1}{6}$ Feet uniformly in that Time, or to have deſcended through $\frac{1}{2}s$ or $16\frac{1}{12}$ Feet in that Time. Whence

$t : s :: \dot{x} : \frac{s\dot{x}}{t} =$ Fluxion of the Space z when $x = t$. wherefore from the general Analogy ($\dot{z} \propto x\dot{x}$) we have $\frac{s\dot{x}}{t} : t\dot{x} :: \dot{z} : x\dot{x}$, and $\dot{z} = \frac{sx\dot{x}}{tt}$, and finding the Fluent, $z = \frac{sx^2}{2tt}$, which needs no Correction (becauſe when $z = 0$, $x = 0$.)

Or thus from the Fluxion and Fluent, $\frac{s\dot{x}}{t} : t\dot{x} :: z : \frac{x^2}{2}$, whence $z = \frac{sxx}{2tt}$.

Or laſtly thus, ſince $\frac{1}{2}s = z$, when $t = x$, therefore from the general Analogy ($z \propto xx$) it will be $\frac{1}{2}s : tt :: z : xx$; whence $z = \frac{sxx}{2tt}$, and thus the ſame Equation is obtained any of theſe Ways.

Ex. 3.

If a Body is projected upwards with a given Velocity a, *to find how far it will aſcend in any Time* x.

Let $z =$ Space, $v =$ it's Velocity; then by Mechanics $\dot{z} \propto v\dot{x}$. Now ſince the firſt Velocity is given, therefore the Space s which would be uniformly diſcribed in a given Time t will be given; wherefore $t : s :: \dot{x} : \frac{s\dot{x}}{t} =$ Moment of the Space $\dot{z}$ at the firſt Inſtant,

FIG. Inſtant, that is when $a=v$. therefore from the general Proportion ($\dot{z} \propto v\dot{x}$) $\frac{s\dot{x}}{t} : a\dot{x} :: \dot{z} : v\dot{x} :: \dot{z} : v\dot{x}$; whence $\dot{z} = \frac{sv\dot{x}}{ta}$. but the Velocity the Body loſes is as the Time, therefore $t : a :: x : \frac{ax}{t} =$ Velocity loſt, whence $v = a - \frac{ax}{t}$, therefore $\dot{z} = \frac{s\dot{x}}{t} - \frac{sx\dot{x}}{tt}$; and the Fluent $z = \frac{sx}{t} - \frac{sx^2}{2tt}$, which needs no Correction.

Hence if x be greater than $2t$, the Body will have deſcended again below the Point it was projected from.

Ex. 4.

To find the Time wherein a given Cylinder of Water will empty itſelf by a Hole at the Bottom.

2. Let $AC = b$, $CE = x$, $AE = b - x$, $t =$ Time of running out with the firſt Velocity, $z =$ Time ſought. Now the Moment of the Quantity running out $\propto \dot{z} \times$ Velocity: but the Velocity is as $\sqrt{CE}$, and the Moment of the Quantity is as the Moment of AE, or $-\dot{x}$, whence $-\dot{x} \propto \dot{z}\sqrt{x}$, therefore $\dot{z} \propto \frac{-\dot{x}}{\sqrt{x}}$; But $b : t :: -\dot{x} : \frac{-t\dot{x}}{b} =$ Moment of the Time at the firſt Inſtant. Therefore (from the general Proportion $\dot{z} \propto \frac{-\dot{x}}{\sqrt{x}}$), $\frac{-t\dot{x}}{b} : \frac{-\dot{x}}{\sqrt{b}} :: \dot{z} : \frac{-\dot{x}}{\sqrt{x}} : \dot{z} : \frac{-\dot{x}}{\sqrt{x}}$, whence $\dot{z} = \frac{-t\dot{x}}{\sqrt{bx}}$, and the Fluent is $z = \frac{-2tx^{\frac{1}{2}}}{\sqrt{b}}$; but in the Point A, $z=0$, and $x=b$, therefore the Fluent corrected (by Prop. XII.) is $z = 2t - \frac{2tx^{\frac{1}{2}}}{\sqrt{b}}$; and when $x = 0$, then the whole Time $z = 2t$.

Ex. 5.

Ex. 5.

To find the Time wherein a given Frustum of a Cone will empty itself by a Hole in the Bottom.

Let the Cone be compleated and put $VG=p$ the Altitude. $TG=h$ the Height of the Frustum, $TS=x$, Circle $CD=d$, $VT=b$, $t=$ Time of running out with the first or greatest Velocity, $z=$ Time sought. Proceeding here as in the last Example, you will find $\dot{z}\sqrt{x}\ \alpha$ Moment of the Quantity $\alpha\ -\dot{x}\times$ Circle $EF\ \alpha$ $\frac{-d\dot{x}}{pp}\times\overline{b+x}^2$. Therefore $\dot{z}\ \alpha\ \frac{-d\dot{x}}{pp\sqrt{x}}\times\overline{b+x}^2$; then $\frac{-t\dot{x}}{h}:\frac{-d\dot{x}}{\sqrt{h}}::\dot{z}:\frac{-d\dot{x}}{pp\sqrt{x}}\times\overline{b+x}^2::\dot{z}:\frac{-d\dot{x}}{pp\sqrt{x}}\times\overline{b+x}^2$, from the general Analogy; therefore $\dot{z}=\frac{-t\dot{x}}{pp\sqrt{hx}}\times\overline{b+x}^2$. And the Fluent is $z=\frac{-t}{pp\sqrt{h}}\times$ $:2bbx^{\frac{1}{2}}+\frac{4}{3}bx^{\frac{3}{2}}+\frac{2}{5}x^{\frac{5}{2}}$; and when corrected, the whole Time $z=t\times\frac{2bb+\frac{4}{3}bh+\frac{2}{5}hh}{pp}$.

And if the Frustum were inverted, the Time would be found to be $t\times\frac{2pp-\frac{4}{3}ph+\frac{2}{5}hh}{bb}$.

SCHOL.

If EF be the double Ordinate in any Curve CA; the Time of running out might have been found the same way, only by substituting the Value of CA or y had from the Nature of the Curve, into the Equation $\dot{z}=\frac{-ty^2\dot{x}}{d\sqrt{hx}}$, and then finding the Fluent.

Ex. 6.

Let ACE be (the Section of) a Wall supporting a Fluid behind it, and joining to the perpendicular Side AC; To find the Curve ADE terminating the other Side of the Wall, so that its Strength may be every where as the Pressure it sustains. FIG. 3. 4.

Let

FIG. Let $AC=b$, $AB=x$, $BD=y$. The Effect which any Number of Particles of the Fluid pressing at B have to break the Wall at C, is as $CB\times$ Number of Particles $\times$ their Force, that is $\propto \overline{b-x}\times\dot{x}\times x$, (because the Number is as $\dot{x}$, and the Pressure or Force as x). And the Sum of all the Forces acting on AB to break C is as the Sum of all the $\overline{b-x}\times x\dot{x}$, that is as the Fluent of $bx\dot{x}-x^2\dot{x}$, and therefore as $\frac{bx^2}{2}-\frac{x^3}{3}$, and when $x=b$, the whole Pressure on AC to break it at C will be $\frac{1}{6}b^3$; therefore the Effects of the Pressure at B and C will be as AB^3 and AC^3. But the Strength of the Wall in B and C is supposed to be as these Forces, and by Mechanics 'tis known to be as BD^2 and CE^2; Therefore $AB^3 : AC^3 :: BD^2 : CE^2$, that is $y^2\propto x^3$; And the Curve ADE is a Semicubical Parabola whose Vertex is A, and therefore convex towards AC.

Ex. 7.

4. *Suppose a Wind to blow against the perpendicular Side AC of the Wall ACE; to find the Curve ADE bounding the other Side, so as the Strength of it be every where as the Force it sustains.*

Let $AB=x$, $BD=y$, $AC=b$. The Force of any Particles at B to break the Wall at C is as $CB\times$ Number of Particles $\propto \overline{b-x}\times\dot{x}$, or $\propto \overline{b-x}\times\dot{x}$; and therefore the whole Force of all the Particles on AB to break the Wall at C is $\propto$ the Fluent of $b\dot{x}-x\dot{x}$ $\propto bx-\frac{x^2}{2}$; therefore the Force to break it at C by all the Particles on AC is $\frac{1}{2}bb$ or as bb, and this must be as the Strength of the Wall or as CE^2; consequently $x^2\propto y^2$ and $x\propto y$, therefore ADE is a right Line.

Ex. 8.

Ex. 8.

Let ABC be a heavy Body, BC a Spring fixt to the Block D: Let AB be close thrust up to C, that the Spring may be close Bent, and fixt thus to the Stock D by a Pin. To find with what Velocity the Body will be projected by the Force of the Spring when the Pin is suddenly plucked out. FIG. 5.

Let $BC=b$ the Length of the Spring in it's natural Position, $CN=x$, $v=$ Velocity of the Point B when it arrives at N, $w=$ Weight of the Body AB, $t=$ Time of describing CN. By Mechanics or the Laws of Motion $\dot{v} \propto \frac{force \times \dot{t}}{Body} \propto$ (by the Nature of a Spring) $\frac{b-x}{w}\dot{t}$: Likewise by the Laws of Motion $\dot{t} \propto \frac{\dot{x}}{v}$ therefore $\dot{v} \propto \frac{b-x}{w} \times \frac{\dot{x}}{v}$, or $v\dot{v} \propto \frac{b\dot{x}-x\dot{x}}{w}$, and finding the Fluent $v^2 \propto \frac{2bx-xx}{w}$, and when $x=b$, $v^2 \propto \frac{bb}{w}$; that is the Square of the Velocity is reciprocally as the Weight of the Body: Consequently, if the Body is projected horizontally the Square of the Distance it is projected to, will also be reciprocally as the Body.

Ex. 9.

Let BC be the Quadrant of a Circle, A the Center; RS parallel to BA. To find the Nature of the Curve DFQ that constantly bissects the Angle made by RF and the Arch FC. 6.

Describe the Circle ne infinitely near FC, and draw npA, and no parallel to CA; then since the Angle $oFn = nFp$, and Side Fn common, and the Angles o, p right, therefore $on = np$.

Let $AS=x$, $SF=y$, AB or $Ap=v$, $AD=b$, and since $on=np$, that is $\dot{x}=\dot{v}$, therefore $x=v$,

Q and

FIG. and the Fluent $x = \dot{v}$, and corrected $x = v - b$, or $b + x = v$. But by Prop. 47. *Eu.* I. $\overline{b+x}^2$ $(= v^2)$ $= yy + xx$, that is $bb + 2bx = yy$; whence the Curve DFQ is a Parabola, whose Latus rectum is $2b$, and Focus the Point A.

Ex. 10.

7. *To find the Time of the Vibration of a Pendulum in an extremely small Arch of a Circle.*

Let the Length of the Pendulum $CB = r$, Cord $AB = c$, $BF = x$, Arch $BE = v$, $Ee = \dot{v}$, $Ff = \dot{x}$, then $BD = \frac{cc}{2r}$, $BG = \frac{cx}{2r}$, $DG = \frac{c}{2r} \times \overline{c - x}$, $EG = \sqrt{cx - \frac{ccxx}{4rr}}$. Also let $t =$ Time of a Body's descending or ascending through the Cord AB, $z =$ Time of descending or ascending through the Arch BE, $\frac{1}{2}t =$ Time of describing AB with the Velocity in B; by the Nature of the Circle $\dot{v} = \frac{c\dot{x}}{2\sqrt{cx - \frac{ccxx}{4rr}}}$.

The Times of describing any Spaces uniformly are as the Spaces directly and the Velocities reciprocally; but since the Pendulum falls from A and is supposed to describe the Arch AB in descending or BA in ascending, therefore the Velocities in B and E are as $\sqrt{DB}$ and $\sqrt{DG}$, or as $\sqrt{AB}$ and $\sqrt{AF}$; therefore $\frac{1}{2}t : \frac{c}{\sqrt{c}} :: \dot{z} : \frac{\dot{v}}{\sqrt{c - x}} :: \dot{z} : \frac{\dot{v}}{\sqrt{c - x}}$; whence $\dot{z} = \frac{t\dot{v}}{2\sqrt{cc - cx}} = \frac{trx^{-\frac{1}{2}}\dot{x}}{2\sqrt{4crr \begin{smallmatrix} -4rr \\ -cc \end{smallmatrix} x + cxx}} =$ (rejecting $cxx - ccx$ as extremely small) $\frac{tx^{-\frac{1}{2}}\dot{x}}{4\sqrt{c - x}}$: And the Fluent (by Form the 10th) is $z = \frac{t}{2r} \times$ Arch of

this

this Circle whose Sine is $r\sqrt{\frac{x}{c}}$, and when $x=c$, then FIG.
$z=\frac{t}{2r}\times$ Quadrant $BH=\frac{t}{4}\times 3.1416$: And $2z$ or the Time of an entire Vibration is $=\frac{1}{2}t\times 3.1416$; Or, which is the same thing, $2z=\frac{3.1416}{2}\times$ Time of descending through twice the Length of the Pendulum. Or putting $s=16\frac{1}{12}$ Feet, $r=$ Feet in the Pendulum, then $2z=\frac{3.1416}{2}\times\sqrt{\frac{2r}{s}}$, in Seconds.

Ex. 11.

To find the meridional Parts for any Latitude. 8.

Let Radius $CA=r$, the given Arch of Latitude $AB=v$, Sine $DB=y$, meridional Parts of $AB=z$. By Construction of Mercator's Chart, as Cosine of the Latitude $(\sqrt{rr-yy})$: Radius (r) : : $(\dot{v}:\dot{z}::)$ $\dot{v}:\dot{z}=\frac{r\dot{v}}{\sqrt{rr-yy}}$, but by the Nature of the Circle $\dot{v}=\frac{r\dot{y}}{\sqrt{rr-yy}}$, whence $\dot{z}=\frac{rr\dot{y}}{rr-yy}$; whence by Form the 6th, $z=\frac{2.30258r}{2}\times\text{Log.}\frac{r+y}{r-y}=2.30258r\times\text{Log.}\sqrt{\frac{r+y}{r-y}}$. But in the Triangle EBF, as EB $(\sqrt{rr-yy})$: Rad. (r) : : EF $(r+y)$: Tangent of the Angle B $=\frac{r\times\overline{r+y}}{\sqrt{rr-yy}}$ $=r\sqrt{\frac{r+y}{r-y}}=$ Cotangent of half the Complement of the Latitude AB, whence $z=2.3025r\times$Log. Cotangent of half the Complement of the Latitude. And by Correction $z=2.302585r\times$Log. Cot. of half the Co. Lat. — $2.30258r\times$Log. Rad. And since the Meridional Parts in the Tables are expressed in Minutes, therefore $z=\frac{2.30258r\times 180\times 60}{3.14159}\times\overline{\text{Log. Cot.}\ \tfrac{1}{2}\text{Co. Lat.}-\text{Log. Rad.}}$

=

F I G. $= 7915,705r \times \text{Log.} \frac{\text{Cot.} \frac{1}{2} \text{Co. Lat.}}{\text{Rad.}} = 7915,705r \times$
$\text{Log.} \frac{\text{Radius}}{\text{Tan.} \frac{1}{2} \text{Co. Lat.}}$.

Cor. 1. Hence the Meridional Parts for the Difference of Latitude of two Places is $= 7915,7r \times$ Difference of the Log. Tangents of half their Complements of Latitude.

Cor. 2. Since as Radius : to meridional Difference of Latitude : : ſo Tangent of the Courſe : to the Difference of Longitude ; and $\frac{\text{Rad.} = 1}{7915,705} = ,000126331$, therefore

As ,000126331 :
To Tangent of the Courſe : :
So the Diff. of the Log. Tangents of half the Comp. of the Lat.
To the Difference of Longitude.

Ex. 12.

9. *To find the Nature of the Curve which a heavy flexible Line will form itſelf into by its Gravity.*

Let the Line be ſuſpended on the two fixt Points *D*, *P*, and diſpoſe itſelf into the Curve *DAP* ; *A* its Vertex, *AQ* its Axis, and *BC* an ordinate.

1. The Part of the Curve *ABD* is kept in its Poſition by a certain Force at *A* acting in Direction *AZ* parallel to the Horizon ; for if the Line be cut through at *A* it will reduce itſelf to a perpendicular Poſition. And this Force acting at *A* is always the ſame whatever Length the Curve be of ; for if the Line be cut through at *B*, and then the Point *B* faſtned to the Plane, it is evident the Force at *A* is neither greater nor leſs ; for the Reſiſtance of the Point *B* does the ſame as the Tenſion of the Line in *B* did before ; and the Force in *A*, or the Tenſion of the Line in *A* muſt remain the ſame.

2. Let

2. Let $a =$ the given Part of the Line, whose Weight is equal to the Tension of the Line in A, and $AC=x$, $BC=y$, $AB=z$, draw the Tangent BS, and $BR=BA$, perpendicular to the Horizon, and RS parallel to it. The Line BA is sustained by three Forces, for its Gravity acts in Direction BR, it is drawn at A in Direction AZ by the Force a, and it is sustained in B by the Tension of the Line in Direction SB; and these three Forces being as BR, RS, and BS, and $BR=z$ by Construction, therefore the Force $a = RS$; whence by similar Triangles $BR\ (z) : RS\ (a) :: bo\ (\dot{x}) : Bo\ (\dot{y}) :: \dot{x} : \dot{y}$, and $a\dot{x}=z\dot{y}$, which is one Property of the Curve.

3. Take $Br=Bb$ the Increment of the Curve, draw rn parallel and Bn perpendicular to BS, then is $nB=Bo$, and $rn=bo$. Since BR is the perpendicular Force or Weight of the Line at B, Br or $\dot{z}$ is the Increment thereof, and therefore $\dot{x}$ is the Increment of the Tension of the Line in B, and $\dot{x}$ is the Fluxion of the Tension, and therefore the Fluent $x =$ Tension or Force acting in Direction nr; But in A where $x = 0$, this Tension $= a$, therefore by Correction. The whole Tension drawing in Direction of the Curve is $a + x$; and this is the Force BS, as was shewn before: Therefore again by similar Triangles $a + x\ (BS) : z\ (BR) :: \dot{z} : \dot{x} :: \dot{z} : \dot{x}$, whence $a\dot{x}+x\dot{x}=z\dot{z}$, and the Fluent $2ax + xx = zz$, which is another Property of the Curve.

4. If the Point B be so taken that the Angle RBS or SBC be half a right Angle, then will AB or z be $= a$: for then $\dot{x}=\dot{y}$, and $z = (\frac{a\dot{x}}{\dot{y}} =)\ a$.

5. Since $\dot{y} = \frac{a\dot{x}}{z} = \frac{a\dot{x}}{\sqrt{2ax+xx}} = \frac{a\dot{z}}{\sqrt{aa+zz}}$; therefore, by Form the 9th, $y = 2.30258\,ax$ Log. $\frac{a+x+\sqrt{2ax+xx}}{a} = 2.30258ax\,\text{Log.}\frac{z+\sqrt{aa+zz}}{a}$, whence the Curve may be easily constructed.

Ex. 13.

Ex. 13.

FIG. 10. *To find the Nature of the Curve BM in which a Body moving (after its fall through AB), it shall descend equal Spaces in equal Times.*

Let $AB=a$, $BP=x$, $PM=y$, now since the Velocities of Bodies are as the Spaces described in equal Times, and the Squares of the Velocities are as the Heights fallen from; therefore $a : a+x ::$ (Square of the Velocity in the Axis at P : to Square of the Velocity in the Curve at $M :: Pp^2 : Mm^2 :: \dot{x}^2 : \dot{x}^2+\dot{y}^2 ::$) $\dot{x}^2 : \dot{x}^2+\dot{y}^2$, and by Division $a : x :: \dot{x}^2 : \dot{y}^2$; therefore $x\dot{x}^2=a\dot{y}^2$, or $x^{\frac{1}{2}}\dot{x}=a^{\frac{1}{2}}\dot{y}$, and finding the Fluent $\frac{2}{3}x^{\frac{3}{2}}=a^{\frac{1}{2}}y$, or $ay^2=\frac{4}{9}x^3$: Therefore the Curve is a semicubical Parabola convex towards BP.

Ex. 14.

11. *If a Body be projected from any Point A parallel to the horizontal Plane BC, and be urged towards that Plane with a Force which is as any Power of its Height above the Plane; to find the Nature of the Curve it will describe.*

Met $AB=r$, $AD=x$, $DP=y$, $v=$ Velocity acquired in falling through AD, $t=$ Time of falling, $f=$ Force in D or P, which suppose to be as BD^n. Now from the Laws of Motion $\dot{v} \propto f\dot{t}$, but $\dot{t} \propto \frac{\dot{x}}{v}$ and $f \propto \overline{r-x}^n$, therefore $\dot{v} \propto \frac{\overline{r-x}^n\dot{x}}{v}$, and $\dot{v} \propto \frac{\overline{r-x}^n\dot{x}}{v}$ or $v\dot{v} \propto \overline{r-x}^n\dot{x}$; and finding the Fluent $\frac{vv}{2} \propto \frac{-\overline{r-x}^{n+1}}{n+1}$, but in A, $x=0$, therefore the Fluent corrected is $vv \propto \frac{r^{n+1}-\overline{r-x}^{n+1}}{n+1}$. But the Fluxion of the Axis is as the Velocity of a descending Body, and

and the Fluxion of the Ordinate is as the Fluxion of the Time, or as a given Quantity C; therefore FIG.

$\dot{x}:\dot{y}::(v=)\sqrt{\frac{r^{n+1}-\overline{r-x}^{n+1}}{n+1}}:b$, whence $\dot{y}=$

$\frac{b\dot{x}}{\sqrt{\frac{r^{n+1}}{n+1}-\frac{\overline{r-x}^{n+1}}{n+1}}}$, or $\dot{y}=\frac{b\dot{x}}{\sqrt{r^{n+1}-\overline{r-x}^{n+1}}}$,

and the Fluent will give the Nature of the Curve.

Case 1. *Let* $n=1$, then $\dot{y}=\frac{b\dot{x}}{\sqrt{2rx-xx}}$; describe the Quadrant AEF; then by the Nature of the Circle, Arch $AE=$ Fluent of $\frac{r\dot{x}}{\sqrt{2rx-xx}}$, therefore take y or $DP=\frac{b}{r}\times$ Arch AE, and P will be in the Curve required. 12.

Case 2. *Let* $n=0$, then $\dot{y}=\frac{b\dot{x}}{\sqrt{x}}$, and $y=2b\sqrt{x}$, or $2bbx=yy$; therefore the Curve will be a Parabola, as is well known.

Case 3. *Suppose* $n=-1$. Here we must have Recourse to the original Process it self, and there we shall have $v\dot{v}\propto\frac{\dot{x}}{r-x}$; describe the Hyperbola HE to the Assymptotes AB, BC; then the Area $ADEH=$ Fluent of $\frac{\dot{x}}{r-x}$, therefore $v\propto\sqrt{ADEH}$, and $\dot{y}=\frac{b\dot{x}}{\sqrt{\text{Area}\,ADEH}}$ for the Nature of the Curve AP. 13.

Case 4. *Let* $n=-2$, then $\dot{y}=b\dot{x}\sqrt{\frac{rr-rx}{x}}$. 14.

Let AE be a Cycloid, $AG=v$, $DE=u$, AGB being the generating Circle; then $u=v+\sqrt{rx-xx}$, and $\dot{u}=\dot{v}+\frac{r-2x}{2\sqrt{rx-xx}}\dot{x}=$ (because $\dot{v}=\frac{r\dot{x}}{2\sqrt{rx-xx}}$,)

$\dot{x}$

FIG. $\dot{x}\sqrt{\frac{r-x}{x}}$. But $\dot{y}=br^{\frac{1}{2}}\times\dot{x}\sqrt{\frac{r-x}{x}}$, whence $y=bu\sqrt{r}$, therefore take $DP=b\sqrt{r}\times DE$, and P will be in the Curve.

Case 5. *Let* $n=-3$; then $\dot{y}=\frac{br\times\overline{r\dot{x}-x\dot{x}}}{\sqrt{2rx-xx}}$, whence $y=br\sqrt{2rx-xx}$, and the Curve is an Ellipsis whose Semi-axis is AB.

Ex. 15.

15. *To know whether a given Curve Line be concave or convex towards its Axis, in any given Point thereof.*

Let AB be the Axis, BC any Ordinate, take $DB=BF$, and draw the Ordinates De, Fo infinitely near BC; draw en and Cg parallel to AB, and produce eC to k. Now by the Nature of Curvature, if the Curve be concave towards the Axis, in the Point C, and the Ordinates increase, then the Point o of the Curve will fall between k and g, and therefore the Increment go is less than nC, whence ok or $go-gk$, that is $go-nC$, will be negative; but ok is the second Moment of the Ordinate, and that is as the second Fluxion when Fk and De approach to, and coincide with BC; therefore if the Curve be concave towards the Axis the second Fluxion of the Ordinate will be
16. negative. And the contrary will happen if the Curve is convex towards the Axis. Wherefore

Let the Axis $AB=x$, Ordinate $BC=y$, then compute the Value of $\ddot{y}$ by the Nature of the Curve, and substitute Numbers for all the Quantities if there be Occasion, then if its Value comes out negative it is concave in that Point, if affirmative it is convex towards the Axis.

Ex. 16.

Ex. 16.

A Line being inflated into the Form of a Curve, and kept in that Form by a Force or Pressure acting perpendicularly upon every Point of the Curve; to find the Proportion between the Tension of the Curve, and the Force acting perpendicularly upon it. FIG. 17.

Let AB, BC be two equal Particles of the Curve, and let the Force acting on the Particle B reduce it to the Position ABC; compleat the Parallelogram $ABCE$, and draw AD, CD, perpendicular to AB, CB; and then the four Points, A, B, C, D, will lie in a Circle. Let AB or $BC = \dot{z}$, AD or $CD = r$: Now by Mechanics the Point B is acted upon by three Forces BA, BC, BE; BA or BC is the Tension of the Curve, and BE is the Force acting perpendicularly against the Curve, therefore these are to one another as BA to BE, or (by similar Triangles) as $\frac{1}{2}\,BD$ to $\dot{z}$. Let now the Points A, C, approach to B and coincide with it, and $\frac{1}{2}\,BD$ becomes $\frac{1}{2}r$, the Radius of Curvature; and $\dot{z}$ is the Particle of the Curve, whereon the Pressure acts: And therefore the Force acting perpendicularly on any Particle of the Curve is to the Tension of the Curve, as that Particle of the Curve is to the Radius of Curvature in that Point.

COR. Therefore if the Particle of the Curve and its Tension be given, the Force acting against that Particle is reciprocally as the Radius of Curvature; that is, directly as the Curvature.

Ex. 17.

EAF is a Curve Line supporting a Fluid; to find the Nature of the Curve. 18.

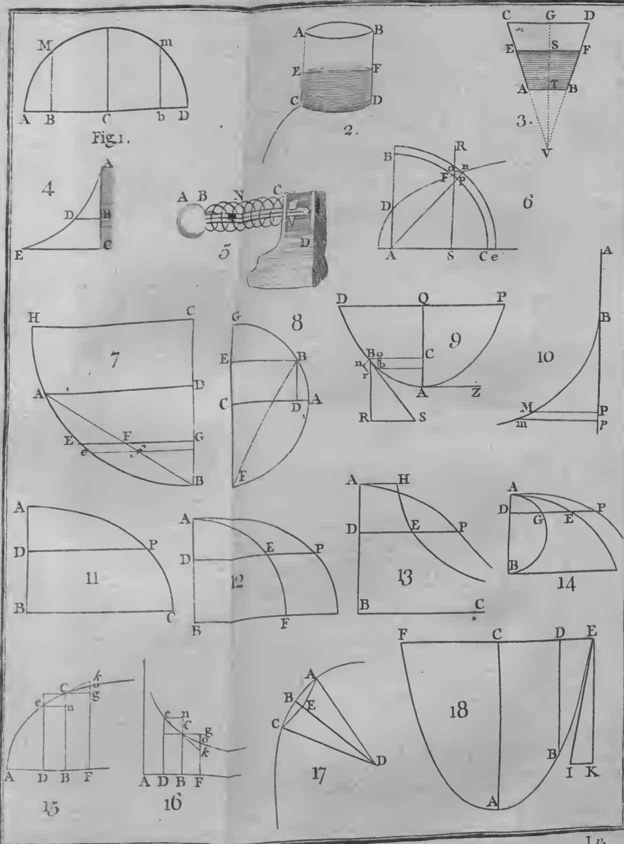
Let the Axis of the Figure $CA = b$, $CD = x$, $DB = y$, $AB = z$, let $\dot{z}$ be given; By reason of the fluidity of the Water, the Tension of the Curve is equal in all Points, and therefore by the foregoing Example, the Pressure at B is reciprocally as the Radius of Curvature

in B. But by the Laws of Hydroſtatics the Preſſure at B is as the Height DB or y, and (by Prob. V. Sect II.) the Radius of Curvature is $\frac{\dot{z}\dot{y}}{\ddot{x}}$, therefore $\frac{1}{y} \propto \frac{\dot{z}\dot{y}}{\ddot{x}}$, and aſſuming the given Quantity $\frac{aa}{2}$, then $\frac{aa}{2y} = \frac{\dot{z}\dot{y}}{\ddot{x}}$, or $\frac{aa\ddot{x}}{2} = y\dot{y}\dot{z}$, and the Fluent $aa\dot{x} = y^2\dot{z}$; but in A, $\dot{x} = \dot{z}$ and $y = b$; therefore the Fluent corrected is $aa\dot{x} - aa\dot{z} = \overline{yy - bb} \times \dot{z}$, or $aa\dot{x} = \overline{aa + yy - bb} \times \sqrt{\dot{x}^2 + \dot{y}^2}$, which reduced gives $\dot{x} = \frac{\overline{aa - bb + yy} \times \dot{y}}{\sqrt{\begin{matrix} 2aabb + 2bb \\ -b^4 - 2aa \end{matrix} y^2 - y^4}}$, for the Equation of the Curve ſought.

COR. Draw EK perpendicular, and IK parallel to ED, and alſo the Tangent EI; then if $EK = p$, $EI = q$, $s =$ Area EAC, which is as its Weight, then (by Mechanics) $p : q :: s : \frac{qs}{p} =$ Tenſion of the Curve, (whence by the foregoing Example) ſince $y\dot{z} =$ Preſſure at B, it will be $y\dot{z} : \frac{qs}{p} : \dot{z} : \frac{\dot{z}\dot{y}}{\ddot{x}}$, and thence $\frac{qs\ddot{x}}{p} = y\dot{y}\dot{z} = \frac{aa}{2}\ddot{x}$, and therefore $\frac{2qs}{p} = aa$.

And let theſe few Examples here ſuffice to ſhew the Application of the Rules to the Calculation of particular Problems. We will now proceed to the Reſolution of more general Problems in Mathematics and Philoſophy.

SECT.



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SECT. II.

The Investigation and Solution of some of the most general and useful Problems in the Mathematics.

PROB. I.

To determine the Maxima and Minima of Quantities.

WHEN a Quantity is required to be the greatest or least possible, under certain Conditions, it is called a Maximum or Minimum; and at that Moment it neither increases nor decreases, and therefore its Fluxion is nothing. Now since any Quantity is a Maximum or Minimum, when its Fluxion is nothing, upon the Supposition of only one variable Quantity therein, and the same is true when any other Quantity alone is supposed variable: Consequently when there are several variable Quantities in the Maximum or Minimum, then each of these Fluxions must be separately equal to nothing. Therefore

1. *Put the constant Quantity m for the Maximum or Minimum required; and get an Equation involving m,*

 by

by help of which and other given Equations, exterminate as many of the variable Quantities as you pleaſe, if there be ſeveral: Then put the remaining Equations into Fluxions, and for each Equation exterminate one Fluxion, till you have but one Equation; in which (brought all to one Side) make the Sum of all the Terms multiplying each particular Fluxion, ſeparately $=0$, *and you will get as many Equations, which, together with thoſe at firſt given, will determine all the unknown Quantities.*

2. *But when the Fluxion of any ſingle Quantity is found* $=0$, *then that Quantity itſelf is either a Maximum or a Minimum, or a ſtanding Quantity. Or if an impoſſible Equation come out, the Quantity will have no Maximum or Minimum but what is infinite. Oftentimes the Equation will have ſeveral Roots, all which muſt be ſeparately tried to ſee which of them will anſwer the Conditions of the Queſtion, and give the Maximum or Minimum required; and this is done by putting the ſingle variable Quantity in the Maximum or Minimum equal to ſeveral ſucceſſive Values expreſſed in Numbers.*

SCHOL. Concerning curvilineal Spaces, it is in Effect the ſame Thing to ſeek the greateſt Area contained under a given Perimeter, as to ſeek a given Area under the leaſt Perimeter. The ſame will hold in Reſpect of Solids and their Surfaces.

Example 1.

To find $xz+yy$ *a Minimum, ſo that* $x+y+z=b$.

Put $xz+y^2=m$, and expunging z, $bx-xx-xy+yy=m$. In Fluxions $b\dot{x}-2x\dot{x}-y\dot{x}-x\dot{y}+2y\dot{y}=0$. And taking the homologous Terms, $b\dot{x}-2x\dot{x}-y\dot{x}=0$, and $2y\dot{y}-x\dot{y}=0$, whence $2x+y=b$, and $x=2y$; and therefore $y=\frac{1}{5}b$, $x=\frac{2}{5}b$, $z=\frac{2}{5}b$.

Or thus;

Since $xz+yy=m$, in Fluxions $x\dot{z}+z\dot{x}+2y\dot{y}=0$, alſo from the Equation $x+y+z=b$, we have $\dot{x}+\dot{y}+\dot{z}=0$; and expunging $\dot{z}$, we get $-x\dot{x}-x\dot{y}+z\dot{x}+2y\dot{y}=0$,

$2y\dot{y}=0$, and therefore $z\dot{x}-x\dot{x}=0$, or $z=x$; and FIG.
$2y\dot{y}-x\dot{y}=0$, or $x=2y$, whence x, y, z are found the ſame as before.

Note, In this Example if x be given there will be a Minimum, but the Maximum is infinite; And if y be given there will be only a Maximum: Therefore in general there is no Maximum or Minimum but what is infinite.

Ex. 2.

To find x the greateſt in this Equation $x^3-ax^2+axy-y^3=0$.

For x write m and the Equation is $m^3-am^2+amy-y^3=0$; In Fluxions $am\dot{y}-3y^2\dot{y}=0$, whence $3y^2=am$, by which and the former Equation m and y are determined.

Ex. 3.

To find a Cylinder of a given Solidity b, with the leaſt whole Surface.

Let $y=$ Altitude, $x=$ Diameter of the Baſe, $c=3.1416$, then $\frac{cx^2y}{4}=b$, and $\frac{cxx}{2}=$ Sum of the Baſes, $cxy=$ convex Surface $=\frac{4b}{x}$; therefore $\frac{cx^2}{2}+\frac{4b}{x}=m$: In Fluxions $cx\dot{x}-\frac{4b\dot{x}}{xx}=0$, which reduced gives $x=\sqrt[3]{\frac{4b}{c}}$, and thence $y=\sqrt[3]{\frac{4b}{c}}=x$.

Ex. 4.

In the Semicircle ABC, to find the rectangle ADB a Maximum. 19.

Let $AC=a$, $AD=x$, $DB=\sqrt{ax-xx}$, then $x\sqrt{ax-xx}=m$, or $ax^3-x^4=m^2$; In Fluxions $3ax^2\dot{x}-4x^3\dot{x}=0$, whence $x=\frac{3}{4}a$.

Ex. 5.

Ex. 5.

FIG. 20. *Given the Base AB and Perpendicular CD, in a Triangle ACB, to find the Angle ACB, the greatest possible.*

Bissect AB in E, and let $CD=p$, $AE=q$, $ED=y$: and by Trigonometry $(CB)\ \sqrt{pp+qq-2qy+yy}$: (Radius) $R::p:\dfrac{pR}{\sqrt{pp+qq-2qy+yy}}=S.\angle B$; and $(AC)\ \sqrt{pp+\overline{q+y}^2}$: $(S.\angle B)\ \dfrac{pR}{\sqrt{pp+\overline{q-y}^2}}$ $::2q:\dfrac{2pqR}{\sqrt{pp+\overline{q+y}^2}\times\sqrt{pp+\overline{q-y}^2}}=S.\angle C$ $=m$, and $p^4+2ppqq+q^4+2p^2y^2-2q^2y^2+y^4=\dfrac{4ppqqR^2}{mm}$: In Fluxions $4ppy\dot{y}-4qqy\dot{y}+4y^3\dot{y}=0$, and $y^3=\overline{qq-pp}\times y$, and one of the Roots is $y=0$; the other Quantity $y^2=qq-pp$ is an impossible Equation when p is bigger than q; wherefore the Point D falls in E.

Ex. 6.

21. *The Point P being given in the Transverse of the Ellipsis ADR; to find PB the nearest Distance to the Curve.*

Let $AC=t$, $CD=c$, $AP=p$, $PR=q$, $PQ=x$; then $QB^2=\dfrac{cc}{tt}\times\overline{pq+qx-px-xx}$, and $PB^2=\dfrac{cc}{tt}\times\overline{pq+qx-px-xx}+xx=m^2$; In Fluxions $\dfrac{ccq\dot{x}-ccp\dot{x}-2ccx\dot{x}}{tt}+2x\dot{x}=0$; reduced $x=\dfrac{cc\times\overline{p-q}}{2\times\overline{tt-cc}}=\dfrac{cc}{tt-cc}\times CP$.

Note, if CP be greater than $\dfrac{tt-cc}{t}$, then x will be greater than PR, which the Nature of the Question will not admit.

Ex. 7.

Ex. 7.

To draw the Line EF to touch the Angle C of the Rectangle ABCD, so that the Part, contain'd between the Sides AB, AD produced, may be a Minimum. FIG. 22.

Let $AD=a$, $AB=b$, $EB=x$, $EA=b+x$, then $AF=\frac{ab+ax}{x}$, and $EF=\sqrt{\overline{b+x}^2+\frac{aa}{xx}\times\overline{b+x}^2}=m$, or $bb+2bx+xx+\frac{aabb}{xx}+\frac{2a\overset{2}{a}b}{x}+aa=m^2$; This in Fluxions is $2b\dot{x}+2x\dot{x}-\frac{2aabb\dot{x}}{x^3}-\frac{2aab\dot{x}}{xx}=0$; reduced is $x^4+bx^3-aabx-aabb=0$, divided by $x+b=0$, one of the Roots, and then $x^3-aab=0$, and $x=\sqrt[3]{aab}$, or from the other Equation $x=-b$.

Or thus:

By similar Triangles $x:a::b:\frac{ba}{x}=DF$, then $EF=\sqrt{aa+xx}+\sqrt{bb+\frac{aabb}{xx}}=m$; In Fluxions $\frac{x\dot{x}}{\sqrt{aa+xx}}-\frac{aabb\dot{x}}{x^3\sqrt{bb+\frac{aabb}{xx}}}=0$; reduced $x^3=aab$.

Ex. 8.

To find a Cone of the greatest Solidity under a given convex Surface and Base b.

Let the Diameter of the Base $=x$, Side $=v$, $c=3{,}1416$, then the Surface $=\frac{cxx}{4}+\frac{cxv}{2}=b$, and Solidity $=\frac{cxx}{12}\sqrt{vv-\frac{1}{4}xx}=m$; And expunging v, $\frac{cxx}{12}\sqrt{\frac{4bb}{ccxx}-\frac{2b}{c}}=m$, or $4bbxx-2bcx^4=144m^2$; In Fluxions, $8bbx\dot{x}-8bcx^3\dot{x}=0$; reduced $x^2=\frac{b}{c}$;

hence

hence $x=\sqrt{\frac{b}{c}}$, $v=\frac{3}{2}\sqrt{\frac{b}{c}}$, and Height $=\sqrt{\frac{2b}{c}}$, and the Height, Baſe, and Side will be as $\surd 2$, 1 and $1\frac{1}{2}$. And the ſame would be true if a Cone of a given Solidity, under the leaſt Surface was required.

Ex. 9.

To find y and x ſuch, that $ay^3-y^2x^2+x^4$ may be a Minimum.

This Equation in Fluxions is $3ay^2\dot{y}-2x^2y\dot{y}-2y^2x\dot{x}+4x^3\dot{x}=0$: And comparing the homologous Terms, $3ay^2\dot{y}-2x^2y\dot{y}=0$, and $4x^3\dot{x}-2y^2x\dot{x}=0$; whence $3ay^2=2x^2y$, and $2x^2=yy$, and therefore $3ay=yy$, and thence $y=0$, or $y=3a$; hence $x^2=0$, or $x^2=\frac{9}{2}a$.

Note, If y be given, the Quantity $ay^3-y^2x^2+x^4$ has a Minimum, but the Maximum is infinite; and if x be given it has a Minimum, but the Maximum is alſo infinite: Therefore if neither be given, it has a Minimum, but no Maximum.

Ex. 10.

To find $xy^2u^3z^4$ a Maximum ſo that $x+y+u+z=b$.

Here $xy^2u^3z^4=m$, and expunging x, $y^2u^3z^4\times\overline{b-y-u-z}=m$, or $b-y-u-z=\frac{m}{y^2u^3z^4}$; In Fluxions $-\dot{y}-\dot{u}-\dot{z}=-\frac{2m\dot{y}}{y^3u^3z^4}-\frac{3m\dot{u}}{y^2u^4z^4}-\frac{4m\dot{z}}{y^2u^3z^5}$; and collecting ſeparately the homologous Quantities, $\dot{y}=\frac{2m\dot{y}}{y^3u^3z^4}$, $\dot{u}=-\frac{3m\dot{u}}{y^2u^4z^4}$, $\dot{z}=\frac{4m\dot{z}}{y^2u^3z^5}$; whence $\frac{m}{y^2u^3z^4}=\frac{y}{2}=\frac{u}{3}=\frac{z}{4}=b-y-u-z$; therefore $u=\frac{3}{2}y$, $z=\frac{4}{3}u=2y$, and therefore $\frac{1}{2}y=b-\frac{9}{2}y$, or $y=\frac{1}{5}b$; hence $x=\frac{1}{10}b$, $y=\frac{2}{10}b$, $u=\frac{3}{10}b$, $z=\frac{4}{10}b$.

Ex. 11.

Ex. 11.

To find a Trapezoid ABCD of a given Area b, whose two Sides and Base AB + BC + CD shall be the least possible. FIG. 23.

Let $BC = x$, $BA = y$, $CD = u$, perpendicular BE or $CF = z$. Then $y + x + u = m$, and $z\sqrt{yy - zz} + z\sqrt{uu - zz} + 2zx = 2b$, or (dividing by z) $\sqrt{yy - zz} + \sqrt{uu - zz} + 2x = \frac{2b}{z}$. Put these Equations into Fluxions, and $\dot{y} + \dot{x} + \dot{u} = 0$, and $\frac{y\dot{y} - z\dot{z}}{\sqrt{yy - zz}} + \frac{u\dot{u} - z\dot{z}}{\sqrt{uu - zz}} + 2\dot{x} = \frac{-2b\dot{z}}{zz}$; and expunging $\dot{x}$, $\frac{y\dot{y} - z\dot{z}}{\sqrt{yy - zz}} + \frac{u\dot{u} - z\dot{z}}{\sqrt{uu - zz}} - 2\dot{y} - 2\dot{u} = \frac{-2b\dot{z}}{zz}$; whence $\frac{y\dot{y}}{\sqrt{yy - zz}} - 2\dot{y} = 0$, and $4z^2 = 3y^2$. Also $\frac{u\dot{u}}{\sqrt{uu - zz}} - 2\dot{u} = 0$, and thence $4z^2 = 3uu$. Therefore $3y^2 = 3u^2$, and $y = u$. Lastly $\frac{-z\dot{z}}{\sqrt{yy - zz}} - \frac{z\dot{z}}{\sqrt{uu - zz}} = \frac{-2b\dot{z}}{zz}$, that is $\frac{z^3}{\sqrt{yy - zz}} + \frac{z^3}{\sqrt{uu - zz}} = 2b$; and expunging z and u, $\frac{z^3}{\frac{1}{2}y} + \frac{z^3}{\frac{1}{2}u} = \frac{4z^3}{y} = \frac{3}{2}y^2\sqrt{3} = 2b$, and $y = 2\sqrt[4]{\frac{bb}{27}} = \frac{2}{3}\sqrt[4]{3bb} = u$, and $x = \frac{b}{z} - \frac{1}{2}\sqrt{yy - zz} - \frac{1}{2}\sqrt{uu - zz} = \frac{2}{3}\sqrt[4]{3bb}$; and $z = \sqrt[4]{\frac{bb}{3}}$. Hence $AB = 2AE = 2FD$, and $AB = BC = CD$: And the Figure is inscribed in a Semicircle, whose Diameter is AD.

Ex. 12.

FIG. 24. *Given the Velocity of a Projectile and its Height AB above the Horizon; To find the Angle of Elevation LAG to throw it to the greatest Distance possible on the Horizontal Plane BC.*

Let $s=AD$ the Space describ'd in a given Time, $d=DE$ the Space describ'd by a falling Body in the same Time; $b=AB$; Let AG be the Line of Projection, AL parallel to BC, $GL=x$, then by the Laws of falling Bodies, $d : b+x :: ss : \frac{bss+ssx}{d} = AG^2$.

Therefore $AL = \sqrt{\frac{bss+ssx}{d} - xx} = BC = m$, or $bss + ssx - dxx = dmm$; in Fluxions, $ss\dot{x} - 2d\dot{x}=0$, whence $x = \frac{ss}{2d}$. Therefore $AG = \frac{s}{2d}\sqrt{4bd+2ss}$, $AL = \frac{s}{2d}\sqrt{4bd+ss}$, $AC = b + \frac{ss}{2d} = CG$; whence the Angle FAC is bissected by the Line of Elevation AG; and $\sqrt{4bd+ss} : s :: (AL : GL ::)$ Rad : Tangent of the Angle of Elevation LAG.

PROB. II.

To find the Logarithms of Numbers.

Logarithms are a certain Set of Numbers, so contriv'd to answer to a Set of Numbers in their natural Order, that the Sum of the Logarithms of any two Numbers shall be the Logarithm of the Product of these Numbers.

Hence therefore, since $1 \times 1 = 1$, the Log. of 1 + Log. 1 = Log. 1, that is the Log. 1 = 0.

Also

Alſo let $AB=C$, then Log. $A+$ Log. $B=$ Log. C; and thence Log. A or Log. $\frac{C}{B}=$ Log. $C-$ Log. B.

Again, Log. $A^2=$ Log. $A+$ Log. $A=2$ Log. A; And Log. $A^3=$ Log. $A+$ Log. $A+$ Log. $A=3$ Log. A; And for the ſame Reaſon Log. $A^n=n\times$ Log. A.

Laſtly, let $AB=1$, then Log. $A+$ Log. $B=$ Log. $1=0$; therefore if the Log. B (or the Log. of a Number greater than 1) be affirmative, the Log. of A (a Number Leſs than 1) is negative.

From what has been ſaid it follows, that if there be a Set of Numbers in geometrical Progreſſion proceeding from 1 both Ways *ad infinitum*; and another Set correſponding thereto, in arithmetic Progreſſion proceeding both Ways from 0; then theſe latter will repreſent the Logarithms of the former, and both will be expreſſed in the following Form,

Numbers, $\frac{1}{n^3}$, $\frac{1}{n^2}$, $\frac{1}{n}$, 1, n, n^2, n^3, n^4, &c.

Logarithms, $-3l$, $-2l$, $-l$, 0, l, $2l$, $3l$, $4l$, &c.

Now let the Number of geometrical Proportionals be increas'd and their Differences decreaſed *ad infinitum*, that the Series may contain all poſſible Numbers; and let the arithmetic Proportionals be in like Manner increaſed; and then the arithmetic Series will alſo contain the Logarithms of all Numbers.

Therefore let n^p be any Number, and pl its Log. then the Increment of that Number will be $n^{p+1}-n^p$, this Increment divided by the Number itſelf, gives $\frac{n^{p+1}-n^p}{n^p}=n-1$; likewiſe the Increment of the Logarithm is $\overline{p+1}\times l-pl$, that is l. Now ſince $n-1$ and l are the ſame for any Number and its Logarithm, therefore the Increment of the Logarithm will always be in a given Ratio to the Increment of the Number

divided by the Number, therefore if we put $x =$ Number, and $z =$ its Logarithm, and aſſume the given Quantity M, we ſhall have $\dot{z} = M \times \frac{\dot{x}}{x}$; and ſince the Fluxions are as the evaneſcent Increments, therefore $\dot{z} = M \times \frac{\dot{x}}{x}$, and the Fluent $z = M \times$ Fluent of $\frac{\dot{x}}{x}$. Therefore,

To find the Logarithm of a Number, divide the Fluxion of the Number propoſed, by the Number it ſelf; and find the Fluent, which multiply by the conſtant Quantity M, and it will give the Logarithm thereof.

SCHOL. When $M = 1$, the Logarithms are called the Hyperbolic Logarithms; in which Caſe, the Fluxion of the Logarithm is equal to the Fluxion of the Number divided by the Number.

Ex. 1.

To find the Log. of bx. Here the Fluxion of the Log. is $M\frac{b\dot{x}}{bx} = M\frac{\dot{x}}{x}$; but the Fluent of $M\frac{\dot{x}}{x} =$ Log. x, therefore the Log. $bx =$ Log. x. But when $x = 1$, then Log. $bx =$ Log. b, and Log. $x = 0$; therefore the Fluent corrected is Log. $bx -$ Log. $b =$ Log. x; that is Log. $bx =$ Log. $b +$ Log. x.

This finds the Logarithm of a Product having the Logarithms of the two Factors given.

Ex. 2.

To find the Logarithm of x^n. Here the Fluxion of its Logarithm $= M \times \frac{nx^{n-1}\dot{x}}{x^n} = M \times \frac{n\dot{x}}{x}$; and taking the Fluent, Log. $x^n =$ Fluent of $M \times \frac{n\dot{x}}{x} = n \times$ Log. x, which needs no Correction (becauſe when $x = 1$, $x^n = 1$, and Log. $x = 0$.)

This

This gives a Rule for finding the Logarithm of any Power or Root of a Quantity, when the Log. of the Quantity is known.

Ex. 3.

To find the Log. $\frac{n+x}{n}$. The Fluxion of the Log. is $= M \times \frac{\dot{x}}{n+x} = M \times : \frac{\dot{x}}{n} - \frac{x\dot{x}}{nn} + \frac{x^2\dot{x}}{n^3} - \frac{x^3\dot{x}}{n^4} +$ &c; therefore the Log. $\frac{n+x}{n} = M \times : \frac{x}{n} - \frac{x^2}{2n^2} + \frac{x^3}{3n^3} - \frac{x^4}{4n^4} +$ &c. If x be negative the Signs of all the odd Powers must be changed.

After the same Manner the Log. $\frac{n}{n-x} = M \times : \frac{x}{n} + \frac{x^2}{2n^2} + \frac{x^3}{3n^3} + \frac{x^4}{4n^4}$ &c.

Cor. The Log. of $b \times \frac{n+x}{n} =$ Log. $b + M \times : \frac{x}{n} - \frac{x^2}{2n^2} + \frac{x^3}{3n^3} - \frac{x^4}{4n^4}$ &c. And the Log. of $b \times \frac{n}{n-x} =$ Log. $b + M \times : \frac{x}{n} + \frac{x^2}{2n^2} + \frac{x^3}{3n^3} + \frac{x^4}{4n^4}$ &c.

Schol. These Series will find the Log. of a Fraction, or the Log. of a Product having the Log. of b. And it finds the Log. of any large Number $n+x$, having the Log. of b, and making $b=n$.

Ex. 4.

To find the Log. of $\frac{n+x}{n-x}$. Its Fluxion is $= 2M \times \frac{n\dot{x}}{nn-xx} = 2M \times : \frac{\dot{x}}{n} + \frac{x^2\dot{x}}{n^3} + \frac{x^4\dot{x}}{n^5} + \frac{x^6\dot{x}}{n^6} +$ &c; Whence the Fluent or Log. $\frac{n+x}{n-x} = 2M \times : \frac{x}{n} + \frac{x^3}{3n^3}$ +

$+\frac{x^5}{5n^5}+\frac{x^7}{7n^7}+$ &c; Or the Log. $\frac{n+x}{n-x}=$ $\frac{2M\times\frac{x}{n}}{1}+\frac{eA}{3}+\frac{eB}{5}+\frac{eC}{7}+\frac{eD}{9}$ &c; where $e=\frac{xx}{nn}$, A, B, C &c are the Numerators of the preceding Terms, with their Signs.

COR. The Log. of $b\times\frac{n+x}{n-x}=\text{Log.}b+\frac{2M\times\frac{x}{n}}{1}+\frac{eA}{3}+\frac{eB}{5}+\frac{eC}{7}$ &c.

SCHOL. The Series in this Example very ſpeedily finds the Log. of a Fraction when x is very ſmall, or n very great.

The Series in this Cor. finds the Log. of a Product, having the Log. b one of the Factors given. Likewiſe it finds the Log. of a large Number $n+x$, having the Log. of b, and making $b=n-x$, and $x=1$ or $\frac{1}{2}$.

Ex. 5.

To find the Log. of n, having the Logarithms of n—x and n+x given.

The Fluxion of the Log. of $\frac{n}{\sqrt{nn-xx}}=\frac{Mx\dot{x}}{nn-xx}$ $=M\times:\frac{x\dot{x}}{nn}+\frac{x^3\dot{x}}{n^4}+\frac{x^5\dot{x}}{n^6}+\frac{x^7\dot{x}}{n^8}:$ &c; whence the Log. $\frac{n}{\sqrt{nn-xx}}=M\times:\frac{x^2}{2n^2}+\frac{x^4}{4n^4}+\frac{x^6}{6n^6}$ &c; but Log. $\frac{n}{\sqrt{nn-xx}}=\text{Log.}\,n-\text{Log.}\sqrt{\overline{nn-xx}}=$ $\text{Log.}\,n-\frac{\text{Log.}\,\overline{n^2-x^2}}{2}=\text{Log.}\,n-\frac{\text{Log.}\,\overline{n+x}}{2}-\frac{\text{Log.}\,\overline{n-x}}{2}$. Therefore $\text{Log.}\,n=\frac{\text{Log.}\,\overline{n+x}+\text{Log.}\,\overline{n-x}}{2}$

+

$+ M\times : \frac{x^2}{2nn} + \frac{x^4}{4n^4} + \frac{x^6}{6n^6} + \&c$: or the Log. n $= \frac{\text{Log.}\, \overline{n+x} + \text{Log.}\, \overline{n-x}}{2} + \frac{\frac{1}{2}Me}{1} + \frac{eA}{2} + \frac{eB}{3} + \frac{eC}{4} + \&c$, where $e = \frac{xx}{nn}$; A, B, C, &c, the Numerators of the preceding Terms.

SCHOL. This Series is very proper for finding the Log. of a large prime Number n, having the Log. of the adjoining Numbers $n-1$ and $n+1$ given, and therefore $x=1$.

Ex. 6.

To find the Log. of n, having the Logarithms of the adjoining Numbers n—1 and n+1 given.

Let $2nn-1=y$, then will $\frac{n}{\sqrt{nn-1}} = \frac{\sqrt{\frac{y+1}{2}}}{\sqrt{\frac{y-1}{2}}} = \sqrt{\frac{y+1}{y-1}}$, the Fluxion of the Log. of $\sqrt{\frac{y+1}{y-1}} = \frac{-\dot{y}}{yy} - \frac{\dot{y}}{y^4} - \frac{\dot{y}}{y^6} - \frac{\dot{y}}{y^8}\ \&c \times M$; therefore the Log. $\sqrt{\frac{y+1}{y-1}} = \frac{1}{y} + \frac{1}{3y^3} + \frac{1}{5y^5} + \frac{1}{7y^7}\ \&c \times M$; and when y is infinite, Log. $\sqrt{\frac{y+1}{y-1}} = 0$, and $\frac{1}{y} + \frac{1}{3y^3} + \frac{1}{5y^7}\ \&c = 0$, therefore still the Log. $\sqrt{\frac{y+1}{y-1}}$ or Log. $\frac{n}{\sqrt{nn-1}} = M\times : \frac{1}{y} + \frac{1}{3y^3} + \frac{1}{5y^5} + \frac{1}{7y^7}\ \&c$; Whence Log. $n = \frac{\text{Log.}\, \overline{n+1} + \text{Log.}\, \overline{n-1}}{2} + M\times : \frac{1}{y} + \frac{1}{3y^3} + \frac{1}{5y^5} + \frac{1}{7y^7} + \&c$.

SCHOL.

SCHOL. This Series is for the ſame Purpoſe, and converges faſter than that in Example the 5th.

Ex. 7.

To find the Logarithm of a large Number n, having the Logarithms of n—x and n+x given.

Put $d = \text{Log.}\,\overline{n+x} - \text{Log.}\,\overline{n-x} = \text{Log.}\,\frac{n+x}{n-x}$.

Then the $\text{Log.}\,n - \text{Log.}\,\overline{n-x} = \text{Log.}\,\frac{n}{n-x} =$

$$\frac{\text{Log.}\,\frac{n+x}{n-x} \times \text{Log.}\,\frac{n}{n-x}}{\text{Log.}\,\frac{n+x}{n-x}} = d \times \frac{\text{Log.}\,\frac{n}{n-x}}{\text{Log.}\,\frac{n+x}{n-x}} =$$

$$\text{(by Ex. 3 and 4)}\ d \times \frac{M \times : \frac{x}{n} + \frac{x^2}{2nn} + \frac{x^3}{3n^3} + \frac{x^4}{4n^4}\ \&c}{2M \times : \frac{x}{n} + \frac{x^3}{3n^3} + \frac{x^5}{5n^5} + \frac{x^7}{7n^7}\ \&c}$$

$= \frac{d}{2} \times : 1 + \frac{x}{2n} + \frac{x^3}{12n^3} + \frac{7x^5}{180n^5} + \&c$, therefore $\text{Log.}\,n = \text{Log.}\,\overline{n-x} + \frac{d}{2} \times : 1 + \frac{x}{2n} + \frac{x^3}{12n^3} + \frac{7x^5}{180n^5} + \&c$: Or thus, ſince $\frac{d}{2} = \frac{\text{Log.}\,\overline{n+x} - \text{Log.}\,\overline{n-x}}{2}$ therefore $\text{Log.}\,n = \frac{\text{Log.}\,\overline{n+x} + \text{Log.}\,\overline{n-x}}{2} + \frac{d}{2} \times : \frac{x}{2n} + \frac{x^3}{12n^3} + \frac{7x^5}{180n^5} + \&c.$

SCHOL. This Series converges far faſter than either of the former Examples.

Ex. 8.

To find the Hyperbolic Logarithm of 10.

Since $\frac{5^{10} \times 2^{30}}{4^{10} \times 10^9}$ or $\frac{5^{10}}{4^{10}} \times \frac{\overline{1024}^3}{\overline{1000}^3} = 10$, therefore

10 Log.

$10 \text{ Log. } \frac{5}{4} + 3 \text{ Log. } \frac{1024}{1000} = \text{Log. } 10.$ By Ex. 4. $\text{Log. } \frac{5}{4} = 2M \times : \frac{1}{9} + \frac{1}{3} \times \frac{1}{9^3} + \frac{1}{5} \times \frac{1}{9^5}$ &c: therefore $10 \text{ Log. } \frac{5}{4} = \frac{20}{9} + \frac{Ae}{3} + \frac{Be}{5} + \frac{Ce}{7} + \frac{De}{9} +$ &c (putting $e = \frac{1}{81}$). Likewise $3 \text{Log. } \frac{1024}{1000} = \frac{18}{253} + \frac{Ae}{3} + \frac{Be}{5} + \frac{Ce}{7}$ &c (putting $e = \frac{9}{64009}$). Here $M = 1$, $A =$ first Term of each Series, and $B, C, D,$ &c, the Numerators of each preceding Term.

The particular Terms of each Series being found will be as follows:

1)	$A =$	2,2222222222222222222222222
3)	$Ae =$	9144947416552354823959 8
5)	$Be =$	67740351233721146844 1
7)	$Ce =$	597357594653625633 5
9)	$De =$	5735943981584882 6
11)	$Ee =$	57938828096817 0
13)	$Fe =$	605248916491 0
15)	$Ge =$	6475914332 8
17)	$He =$	70543729 1
19)	$Ie =$	779235 4
21)	$Ke =$	8704 0
23)	$Le =$	98 1
25)	$Me =$	1 1
	$10 \text{ Log. } \frac{5}{4} =$	2,2314355131420975576629507

	$A =$	,0711462450592885375494071
3)	$Ae =$	333451132149956432 14
5)	$Be =$	2813098335561819
7)	$Ce =$	282525950815
9)	$De =$	30896931
11)	$Ee =$	3554
	$3 \text{ Log. } \frac{1024}{1000} =$	,0711495798519481263550404

H. Log. 10 = 2.30258509299404568401799 11 &c.

 COROL.

COROL. Hence therefore the Number M, mentioned before and made Use of in the foregoing Examples, will be known for the common Logarithms. For since the common Logarithm of 10 is 1, therefore $2.302585 \&c \times M = 1$. Whence $M = \frac{1}{2.302585 \&c} = ,43429448190325182765112 89 \&c.$

After the same Manner may *Brig*'s or the common Logarithms be found, since we know the Number M. For let the former Series or $10 \text{ Log. } \frac{5}{4} = P$; the latter Series or $\text{Log. } \frac{2^{30}}{10^9} = Q$; then the $\text{Log. } 2 = \frac{10 - MP}{30}$ or $\frac{9 + MQ}{30}$, whence is had the Logarithms of 4 and 8; likewise the $\text{Log. } 5 = \frac{20 + MP}{30}$. And thus,

The Log. of $7 = \text{Log. } 8 \times \frac{7}{8} = \text{Log. } 8 - \frac{\frac{4}{30}M}{1} + \frac{\frac{4}{900}A}{3} + \frac{\frac{4}{900}B}{5} + \frac{\frac{4}{900}C}{7} + \frac{\frac{4}{900}D}{9} \&c$, by Example the 4th. And likewise

$\text{Log. } 3 = \text{Log. } 2 \times \frac{3}{2} = \text{Log. } 2 + \frac{\frac{4}{10}M}{1} + \frac{\frac{4}{100}A}{3} + \frac{\frac{4}{100}B}{5} + \frac{\frac{4}{100}C}{7} \&c$; from whence is had the Log. of 6 and 9.

Or the $\text{Log. } 9 = \text{Log. } 10 \times \frac{9}{10} = 1 - \frac{\frac{1}{10}M}{1} + \frac{\frac{1}{10}A}{2} + \frac{\frac{1}{10}B}{3} + \frac{\frac{1}{10}C}{4} \&c$, and $\text{Log. } 11 = \text{Log. } 10 \times \frac{11}{10} = 1 + \frac{\frac{1}{10}M}{1} - \frac{\frac{1}{10}A}{2} - \frac{\frac{1}{10}B}{3} - \frac{\frac{1}{10}C}{4} - \&c.$ Both these by Example the 3d. And so for others.

SCHOL. The Rules in Example the 3d and 4th are the only two Methods by which the Logarithm of a Number can be found without the Help of other given Logarithms.

Logarithms. And therefore to find the Log. of a great Number thereby, we must take two or more such fractional Numbers, that the Product of some Powers of these Numbers may make the Number whose Logarithm is sought: And then there will be had two or more Series, which will give the required Logarithm; in the same Manner as is shewn in Example 8 for the Log. of the Number 10.

PROB. III.

To draw Tangents to Curves.

1. In all geometrical Curves, where there is given the Relation of the Absciffa and Ordinate. Let AP FIG.
$=x$, $PM=y$, and draw mp parallel and infinitely 25.
near to MP, and MR parallel to AP, and draw the Tangent MT, and let Pp or $MR=\dot{x}$, $Rm=\dot{y}$. Then by the similar Triangles TPM and MRm, $\dot{y} : \dot{x} :: y : \frac{y\dot{x}}{\dot{y}} = PT$ the Subtangent, that is $\frac{y\dot{x}}{\dot{y}} = PT$.

Therefore, by Help of the Equation of the Curve, exterminate $\dot{x}$ or $\dot{y}$ out of the Quantity $\frac{y\dot{x}}{\dot{y}}$, and you will get the Value of the Subtangent.

2. *In mechanical or transcendent Curves refer'd to an Axis, the Subtangent $\frac{y\dot{x}}{\dot{y}}$ may be clear'd of the Fluxions by the Equation of the Curve, by the Help of those of the known Curve it is related to.*

3. *And in any Curve QM refer'd to a fixed Point A;* 26.
draw AT perpendicular to AM, and let $AM=y=AR$, $Rn=\dot{y}$, $RM=\dot{x}$; and by similar Triangles $\frac{y\dot{x}}{\dot{y}}$ or $\frac{y\dot{x}}{\dot{y}} = AT$ the Subtangent, out of which the Fluxions may be exterminated as before.

FIG. There are several other Methods of drawing Tangents, but this is the most general and easy.

Example 1.

27. *To draw a Tangent to the Circle.* Let $AP = x$, $PM = y$, Radius $AC = r$; then will $2rx - xx = yy$, whence $2r\dot{x} - 2x\dot{x} = 2y\dot{y}$, and $\dot{x} = \frac{y\dot{y}}{r-x}$. Therefore $PT = \frac{y\dot{x}}{\dot{y}} = \frac{yy}{r-x} = \frac{2rx-xx}{r-x}$.

Ex. 2.

28. *To draw a Tangent to the Ellipsis.* Let transverse $AB = 2a$, Latus rectum $= b$, $AP = x$, $PM = y$. Then by the Nature of the Curve $\frac{2ayy}{b} = 2ax - xx$, and in Fluxions $\frac{4ay\dot{y}}{b} = 2a\dot{x} - 2x\dot{x}$, or $\dot{y} = \frac{ab\dot{x} - bx\dot{x}}{2ay}$, consequently $PT = \frac{y\dot{x}}{\dot{y}} = \frac{2ayy}{ba-bx} = \frac{2ax-xx}{a-x}$.

Ex. 3.

25. *To draw Tangents to all sorts of Parabolas.* Let $AP = x$, $PM = y$, and $x = y^m$, putting the Parameter $= 1$. Then $\dot{x} = my^{m-1}\dot{y}$, therefore $PT = \frac{y\dot{x}}{\dot{y}} = my^m = mx$. And therefore in the common Parabola, where $m = 2$, $TP = 2x = 2AP$.

Ex. 4.

25. *To draw a Tangent to an Hyperbola.* Let $AP = x$, $PM = y$, Transverse $= 2a$, latus rectum $= b$. Then the Property of the Figure is, $\frac{2ayy}{b} = 2ax + xx$; in Fluxions $\frac{4ay\dot{y}}{b} = 2a\dot{x} + 2x\dot{x}$ or $\dot{y} = \frac{a\dot{x} + x\dot{x}}{2ay}b$; therefore $PT = \frac{y\dot{x}}{\dot{y}} = \frac{2ayy}{ba+bx} = \frac{2ax+xx}{a+x}$.

Ex. 5.

Ex. 5.

Let MB be an Hyperbola between the Assymptotes; Put $AP=x$, $PM=y$, and $aa=xy$; then $x\dot{y}+y\dot{x}=0$, or $\dot{x}=\frac{-x\dot{y}}{y}$; whence $PT=\frac{y\dot{x}}{\dot{y}}=-x$, and the negative Sign shews that T lies on the contrary Side of PM with A. FIG. 29.

Ex. 6.

To draw a Tangent to the Cissoid AM. Let $AP=x$, $PM=y$, $AB=a$, BC the Assymptote. Then by the Nature of the Curve $ay^2-xy^2=x^3$; whence $2ay\dot{y}-2xy\dot{y}-y^2\dot{x}=3x^2\dot{x}$, or $\dot{x}=\frac{2ay-2xy}{yy+3xx}\dot{y}$, whence $PT=\frac{y\dot{x}}{\dot{y}}=\frac{2ayy-2xyy}{y^2+3x^2}$, or expunging yy, $PT=\frac{2ax-2xx}{3a-2x}$. 30.

Ex. 7.

To draw a Tangent to the Conchoid. Let $CA=b$, $AV=a$, $AP=x$, $PM=y$. By the Nature of the Curve, $\overline{b+x}\sqrt{aa-xx}=xy$, whence $\dot{x}\sqrt{aa-xx}-\frac{x\dot{x}\times\overline{b+x}}{\sqrt{aa-xx}}=x\dot{y}+y\dot{x}$; whence $PT=\frac{y\dot{x}}{\dot{y}}=\frac{yx\sqrt{aa-xx}}{aa-2xx-bx-y\sqrt{aa-xx}}=\frac{\overline{b+x}\times\overline{aax-x^3}}{-baa-x^3}$; and being negative, it lies on the contrary Side of PM. 31.

Otherwise thus;

Let $AP=v$, $AM=y$, $PM=a$, $AH=b$, $MR=\dot{x}$; Then by the Nature of the Curve $v+a=y$, and $\dot{v}=\dot{y}$; and by similar Triangles $Pp=\frac{v\dot{x}}{y}$, and $\dot{v}=\frac{v\times Pp}{b}=\frac{v^2\dot{x}}{by}$, and thence $\dot{x}=\frac{by\dot{y}}{vv}$ or $\dot{x}=\frac{by\dot{y}}{vv}$; therefore $AT=\frac{y\dot{x}}{\dot{y}}=\frac{byy}{vv}$. 32.

Ex. 8.

Ex. 8.

FIG. 25. *Let AM be the Catenary,* $AP=x$, $PM=y$, $AM=z$, by the Nature of the Curve $zz=2ax+xx$, whence $z\dot{z}=a\dot{x}+x\dot{x}$, and $\dot{z}=\frac{a+x}{z}\dot{x}$; but $\dot{y}^2=\dot{z}^2-\dot{x}^2=\frac{\overline{a+x}^2}{zz}\dot{x}^2-\dot{x}^2=\frac{\overline{a+x}^2-zz}{zz}\dot{x}^2=$ (expunging zz) $\frac{aa\dot{x}^2}{zz}$, and $\dot{y}=\frac{a\dot{x}}{z}$; therefore $PT=\frac{y\dot{x}}{\dot{y}}=\frac{zy}{a}$.

Ex. 9.

33. *To draw a Tangent to the Cycloid.* Let $AP=x$, $PM=y$, $AC=2a$, $PB=u$, Arch $AB=z$. By the Nature of the Figure $y=z+u$, and thence $\dot{y}=\dot{z}+\dot{u}$, but from the Circle $\dot{z}=\frac{a}{u}\dot{x}$, and $\dot{u}=\frac{a-x}{u}\dot{x}$; whence $PT=(\frac{y\dot{x}}{\dot{y}}=\frac{y\dot{x}}{\dot{z}+\dot{u}}=)\ \frac{uy}{2a-x}=\frac{AP}{PB}y$.

Ex. 10.

34. *To draw a Tangent to the Quadratrix AMB.* Let $AC=b$, $CB=r$, $AP=x$, $PM=y$, $DN=z$, $NI=s$, $CM=v$, $NQ=t$, $RM=u$. Now by the Nature of the Curve $bz=ry$, therefore $b\dot{z}=r\dot{y}$, and by the Property of the Circle $r\dot{s}=t\dot{z}$; whence $\dot{z}=\frac{r\dot{y}}{b}=\frac{r\dot{s}}{t}$, and $\dot{s}=\frac{t\dot{y}}{b}$. By similar Triangles $ty=su$, and thence $\dot{t}y+y\dot{t}=u\dot{s}+s\dot{u}$, that is (because $u=b-x$ or $\dot{u}=-\dot{x}$, and $rr-ss=tt$ or $\dot{t}=\frac{-s\dot{s}}{t}$) $t\dot{y}-\frac{sy}{t}\dot{y}=(u\dot{s}-s\dot{x}=)\ \frac{tu\dot{y}}{b}-s\dot{x}$: whence $\dot{x}=\frac{tu\dot{y}}{bs}+\frac{y\dot{y}}{b}-\frac{t\dot{y}}{s}$. Therefore $PT=\frac{y\dot{x}}{\dot{y}}=\frac{tuy}{bs}+\frac{yy}{b}-\frac{ty}{s}=$ (because $u=\frac{ty}{s}$) $\frac{u^2+y^2}{b}-u=\frac{v^2}{b}-u$.

Ex. 11.

Ex. 11.

Let AS be any given Curve, and AM another Curve, such that $AP =$ Arch AS, and Ordinate $PM = AQ$, to draw the Tangent TM. FIG. 35.

Let $AP=x$, $PM=y$, $AQ=v$, Tangent $SR=s$, $QR=t$, $AS=z$. By the Nature of the Figure $x=z$, and $v=y$, therefore $\dot{x}=\dot{z}$ and $\dot{v}=\dot{y}$; and by similar Triangles $\dot{v} = \frac{QR}{SR}\dot{z} = \frac{t\dot{x}}{s}$; therefore $PT = (\frac{y\dot{x}}{\dot{y}} = \frac{y\dot{x}}{\dot{v}} =)\ \frac{sy}{t}$.

Ex. 12.

Let BM be an exponential Curve, $AP=x$, $PM=y$, and $a^x=y$. Let A, Y be the hyperbolic Logarithms of a, y; then $xA=Y$, and $A\dot{x}=\dot{Y}=\frac{\dot{y}}{y}$, whence $PT=\frac{y\dot{x}}{\dot{y}}=\frac{1}{A}$ an invariable Quantity: Therefore BM is the Logarithmic Curve. 36.

Ex. 13.

Let the Nature of an exponential Curve be expressed by this Equation $x^x=y$; Let X, Y be the hyperbolic Logarithms of x, y; then $xX=Y$, and $x\dot{X}+X\dot{x}=\dot{Y}$; but $\dot{X}=\frac{\dot{x}}{x}$, and $\dot{Y}=\frac{\dot{y}}{y}$, (by Prob. II.) therefore $\dot{x}+X\dot{x}=\frac{\dot{y}}{y}$, or $y\dot{x}+yX\dot{x}=\dot{y}$, whence $PT=\frac{y\dot{x}}{\dot{y}} = \frac{y\dot{x}}{y\dot{x}+Xy\dot{x}}=\frac{1}{1+X}$. 36.

Ex. 14.

To draw a Tangent to Archimedes's Spiral. Let Radius $AP=r$, Arch $QP=z$, $AM=y$, c a given Line. By the Nature of the Figure $rz=cy$, and thence $\dot{z}=\frac{c\dot{y}}{r}$, and by similar Triangles $\dot{x}=\frac{y\dot{z}}{r}=\frac{cy}{rr}\dot{y}$, whence $AT=\frac{y\dot{x}}{\dot{y}}=\frac{cyy}{rr}$. 37.

Ex. 15.

Ex. 15.

FIG. 38. *To draw a Tangent to the reciprocal ſpiral.* Let Radius $BA=a$, Arch $BC=b$, $BP=z$, $AM=y$, $Rn=\dot{x}$. The Equation of the Curve is $ab=zy$, and therefore $z\dot{y}+y\dot{z}=0$; and by ſimilar Triangles $y\dot{z}=-a\dot{x}$, or $y\dot{z}=-a\dot{x}$, whence $z\dot{y}-a\dot{x}=0$, or $\dot{x}=\frac{z\dot{y}}{a}$. Therefore $AT=\frac{y\dot{x}}{\dot{y}}=\frac{zy}{a}=b$ a given Quantity.

Ex. 16.

39. *Let AMD be a ſpiral of ſuch a kind, that putting AM or $AC=y$, Arch $MC=v$, $AD=a$ a given Line, and $av^2=y^3$.* Let Arch $BE=z$, and by the ſimilar Sectors AMC, AEB, $av=yz$, whence expunging v, $z^2=ay$, alſo rM or $\dot{x}=\frac{y\dot{z}}{a}$, or $\dot{x}=\frac{y\dot{z}}{a}$. And $2z\dot{z}=a\dot{y}$: whence $\dot{x}=\frac{y\dot{y}}{2z}=\frac{z\dot{y}}{2a}$, and therefore $AT=(\frac{y\dot{x}}{\dot{y}}=\frac{yz}{2a}=\frac{av}{2a}=)\ \frac{1}{2}v$.

PROB. IV.

To find the Point of Inflexion or contrary Flexure of a given Curve.

The Point of Inflexion or contrary Flexure is that Point which ſeparates the convex from the concave Part of the Curve.

40. In the Curve AM any way related to the Axis AP, let $AP=x$, Ordinate $PM=y$; draw the Ordinate pm parallel and infinitely near PM, and Mr parallel to AP. Now the Ratio of Mr to rm is the greateſt or leaſt poſſible in the Point of contrary Flexure, that

that is $\frac{Mr}{mr}$ or $\frac{\dot{x}}{\dot{y}}$ is a Maximum or Mininum, whence (by Prob. I.) its Fluxion $\frac{\dot{y}\ddot{x}-\dot{x}\ddot{y}}{\dot{y}^2}=0$, or $\dot{y}\ddot{x}-\dot{x}\ddot{y}=0$; hence if $\ddot{x}=0$, then $\ddot{y}=0$, or if $\ddot{y}=0$, then $\ddot{x}=0$.

In a Curve BM refer'd to the fixed point P, take Mm infinitely small, and draw Pm, and Mr perpendicular to it, draw the Tangent MT and PT perpendicular to it; and let $PM=y$, Curve $BM=z$, $Mr=\dot{x}$, $mr=\dot{y}$. Then, by similar Triangles, $PT=\frac{y\dot{x}}{\dot{z}}=\frac{y\dot{x}}{\dot{z}}$. But in the Point of Inflexion the perpendicular PT is a Maximum or Minimum, therefore its Fluxion or the Fluxion of $\frac{y\dot{x}}{\dot{z}}=0$, that is $\frac{y\dot{z}\ddot{x}+\dot{x}\dot{y}\dot{z}-y\dot{x}\ddot{z}}{\dot{z}^2}=0$, or $y\dot{z}\ddot{x}-y\dot{x}\ddot{z}+\dot{x}\dot{y}\dot{z}=0$. Out of which any of the Fluxions may be exterminated by help of the Equation $\dot{z}^2=\dot{x}^2+\dot{y}^2$ and its Fluxion, making any of the Fluxions invariable, or the second Fluxion $=0$. Hence therefore to find the Point of Inflexion, the rule is, FIG. 41.

1. *In Curves referred to an Axis, let $AP=x$, $PM=y$, put the Equation of the Curve into Fluxions, and again put the resulting Equation into Fluxions, making both $\dot{x}$ and $\dot{y}$ given quantities.* 40.

2. *In Curves related to a Pole or given Point P; let $PM=y$, $BM=z$, and Mm, mr, Mr will be as $\dot{z}$, $\dot{y}$, $\dot{x}$. Put the Equation of the Curve into Fluxions, and the resulting Equation into Fuxions again, writing 1 for some first Fluxion each Operation: Then you will determine the other Fluxion which must be substituted into one of the following equations that contains that other Fluxion. And then you will get the Value of y.* 41.

FIG.

If $\dot x = 1$. *then* $y = \frac{1+\dot y^2}{\ddot y} = \frac{\dot z\sqrt{\dot z^2-1}}{\ddot z}$.

If $\dot y = 1$. $y = \frac{\dot x+\dot x^3}{-\ddot x} = \frac{\dot z^3-\dot z}{-\ddot z}$.

If $\dot z = 1$. $y = \frac{1-\dot y^2}{\ddot y} = \frac{\dot x\sqrt{1-\dot x^2}}{-\ddot x}$.

Example 1.

Let the Equation of the Curve be $ax^2 = x^2y + aay$. This in Fluxions is $2ax\dot x = 2yx\dot x + x^2\dot y + aa\dot y$; which put again into Fluxions, $2a\dot x^2 = 2y\dot x^2 + 2x\ddot x\dot y + 2x\dot x\dot y = 2y\dot x^2 + 4x\dot x\dot y$, and expunging $\dot y$, $a\dot x^2 = y\dot x^2 + 2x\dot x \times \frac{2ax-2yx}{aa+xx}\dot x$; that is by Reduction $aa + xx = 4xx$; whence $x = a\sqrt{\frac{1}{3}}$.

Ex. 2.

42. *Let BM be Nichomedes's Concoid.* $AB=a$, $AC=b$; then its Nature is $yx = \overline{b+x}\sqrt{aa-xx}$. In Fluxions $y\dot x + x\dot y = \dot x\sqrt{aa-xx} - \frac{bx\dot x + x^2\dot x}{\sqrt{aa-xx}}$; and expunging y, $\frac{-x^3-aab}{xx\sqrt{aa-xx}}\dot x = \dot y$, and this again in Fluxions is $\frac{2a^4b - a^2x^3 - 3aabxx}{\overline{aax^3-x^5}\sqrt{aa-xx}}\dot x^2 = 0$, and reduced $x^3 + 3bxx = 2aab$.

Ex. 3.

43. *Let AFK be a Cycloid of such a kind, that arch AD: DM :: ADB : BK.* Let $AB=2r$, $ADB=a$, $BK=b$, $PD=z$, $AD=u$; then $y = z + \frac{bu}{a}$, and $\dot y = \dot z + \frac{b\dot u}{a}$; by the Property of the Circle $\dot z = \frac{r-x}{\sqrt{2rx-xx}}\dot x$, and $\dot u = \frac{r\dot x}{\sqrt{2rx-xx}}$, whence $\dot y = \frac{ar-ax+rb}{a\sqrt{2rx-xx}}\dot x$; in Fluxions $\frac{brx-arr-brr}{\overline{2rx-xx}|^{\frac{3}{2}}}\dot x^2 = 0$, reduced $x = r + \frac{ar}{b}$.

Ex. 4

Ex. 4.

Let BM be the helicoid Parabola, Radius PB or FIG.
$PE=r$, arch $BE=v$, $PM=y$; then by the Nature 44.
of the Curve $av=\overline{r-y}^2$, whence $a\dot{v}=-2\dot{y}\times\overline{r-y}$, but by similar Sectors $\dot{v}=\frac{r\dot{x}}{y}$, or $\dot{v}=\frac{r\dot{x}}{y}$; whence $ra\dot{x}=-2y\dot{y}\times\overline{r-y}$; and $\dot{y}=\frac{ar}{2y\times\overline{y-r}}$ (putting $\dot{x}=1$); and $\ddot{y}=\frac{-ar\dot{y}}{2yy}\times\frac{2y-r}{\overline{y-r}^2}$, whence $y=\frac{1+\dot{y}^2}{\ddot{y}}=\frac{4yy\times\overline{y-r}^2+aarr}{-2ar\dot{y}\times\overline{2y-r}}$, and expunging $\dot{y}$, $4y^2\times\overline{y-r}^2+aarr=2ary\times\overline{2y-r}\times\frac{-ar}{2yy-2ry}$, that is $4yy\times\overline{y-r}^3+aarr\times\overline{y-r}+aarr\times\overline{2y-r}=0$; which reduced gives $4y^5-12ry^4+12r^2y^3-4r^3y^2+3r^2a^2y-2r^3a^2=0$.

Ex. 5.

Suppose BM to be a sort of Spiral, PT or $PR=a$, 45.
Arch $RT=v$, and let $a^3=vyy$ express its Nature; then $2vy\dot{y}+yy\dot{v}=0$, but by similar Triangles $\dot{v}=\frac{a\dot{x}}{y}$, whence $2vy\dot{y}+ay\dot{x}=0$, and $\dot{x}=\frac{-2v}{a}$, putting $\dot{y}=1$; and expunging v, $\dot{x}=\frac{-2aa}{yy}$, and $\ddot{x}=\frac{4aa}{y^3}$; therefore $y=\frac{\dot{x}+\dot{x}^3}{-\ddot{x}}=\frac{1}{2}y+\frac{2a^4}{y^3}$, or $y^4=4a^4$, whence $y=a\sqrt{2}$.

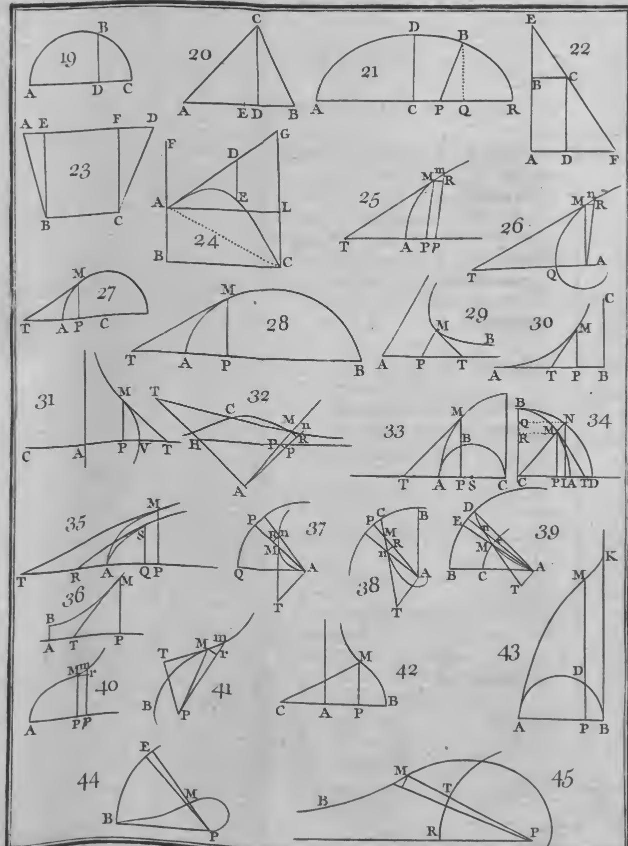
PROB. V.

To find the Radius of Curvature in Curves.

FIG. 46. Let Amq be a Curve referred to the Axis AE, and suppose C to be the Center, and Cm the Radius of Curvature in any Point m, and describe the equicurve Circle Dm coinciding with the Curve in mn. Draw DC, mr parallel, and AF perpendicular to AB, and produce mB to H, and draw nbb parallel and infinitely near mBH; and call the Abscisa AB, x; perpendicular ordinate Bm, y; Am, z; mC, r; DF, b; AF or BH, c; mr, $\dot{x}$; rn, $\dot{y}$; and mn, $\dot{z}$; then $CH=r-b-x$.

By the Nature of the Circle $DH\times\overline{2DC-DH} = Hm^2$, that is $2rb+2rx-bb-2bx-xx = cc+2cy+yy$; which put into Fluxions $r\dot{x}-b\dot{x}-x\dot{x} = c\dot{y}+y\dot{y}$; which put into Fluxions again, $r\ddot{x}-b\ddot{x}-x\ddot{x}-\dot{x}^2=c\ddot{y}+y\ddot{y}+\dot{y}^2$; therefore $\overline{r-b-x}\times\ddot{x}-\ddot{y}\times\overline{c+y}=\dot{x}^2+\dot{y}^2=\dot{z}^2$. But by similar Triangles $r-b-x=\frac{r\dot{y}}{\dot{z}}=\frac{r\dot{y}}{\dot{z}}$, and $c+y = \frac{r\dot{x}}{\dot{z}}=\frac{r\dot{x}}{\dot{z}}$, whence the former Equation becomes $\frac{r\dot{y}\ddot{x}}{\dot{z}}-\frac{r\dot{x}\ddot{y}}{\dot{z}}=\dot{z}^2$; which reduced gives $r=\frac{\dot{z}^3}{\dot{y}\ddot{x}-\dot{x}\ddot{y}}$ for the Radius of Curvature of the Circle Dm or of the Curve Am in the Point m.

Now since we are at Liberty to take any one of the Fluxions $\dot{x}$, $\dot{y}$, $\dot{z}$, as invariable; we can, by the Help of this Equation $\dot{z}^2=\dot{x}^2+\dot{y}^2$ and its Fluxion, exterminate either of the other indetermin'd Fluxions out of the Value of r, and by that Means find several different Forms for the Radius of Curvature, to suit different Cases. And the same Way will several Values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II. p.

Values of mH, CH, be determin'd. Thus we shall find when $\dot{x}$ is given, the Radius $= \frac{\dot{z}^3}{-\dot{x}\ddot{y}} = \frac{\dot{z}^2\dot{y}}{-\dot{x}\ddot{z}}$; when $\dot{y}$ is given, the Radius $= \frac{\dot{z}^3}{\dot{y}\ddot{x}} = \frac{\dot{z}^2\dot{x}}{\dot{y}\ddot{z}}$; and if $\dot{z}$ be given, the Radius $= \frac{\dot{z}\dot{y}}{\ddot{x}} = \frac{\dot{z}\dot{x}}{-\ddot{y}}$. FIG.

Again in Spirals or Curves refer'd to a fixt Point B, as AMn. Let C be the Center, and CM the Radius of Curvature in any Point M; and describe the Circle DMn coinciding with the Curve in the infinitely small part Mn. Draw Bn infinitely near BM, and CD, Mr perpendicular to BM; and call BM, y; Curve AM, z; Radius CM, r; DE, v; Ee, $\dot{v}$; Mr, $\dot{x}$; Mn, $\dot{z}$; rn, $\dot{y}$. By the similar Triangles MEC and Mnr, $ME = \frac{r\dot{x}}{\dot{z}} = \frac{r\dot{x}}{\dot{z}}$, and $EC = \frac{r\dot{y}}{\dot{z}} = \frac{r\dot{y}}{\dot{z}}$, therefore $v = r - \frac{r\dot{y}}{\dot{z}}$, and thence $\dot{v} = \frac{r\dot{y}\ddot{z} - r\dot{z}\ddot{y}}{\dot{z}^2}$: And by the similar Triangles BEe, BMr, it is $BE : BM :: Ee : Mr$, that is $y - \frac{r\dot{x}}{\dot{z}} : y :: (\dot{v} : \dot{x} :: \dot{v} : \dot{x} ::) \frac{r\dot{y}\ddot{z} - r\dot{z}\ddot{y}}{\dot{z}^2} : \dot{x}$; which multiply'd and reduced gives $r = \frac{y\dot{x}\dot{z}^2}{\dot{z}\dot{x}^2 + y\dot{y}\ddot{z} - y\dot{z}\ddot{y}}$, for the Radius of Curvature of the Circle DM, or of the Curve AM in M. Hence also $ME = \frac{y\dot{x}^2\dot{z}}{\dot{z}\dot{x}^2 + y\dot{y}\ddot{z} - y\dot{z}\ddot{y}}$, and $EC = \frac{y\dot{y}\dot{x}\dot{z}}{\dot{z}\dot{x}^2 + y\dot{y}\ddot{z} - y\dot{z}\ddot{y}}$. 47.

Now since we can make any one of the Fluxions $\dot{x}$, $\dot{y}$, $\dot{z}$, invariable; we can expunge either of the rest and its Fluxion by the Help of this Equation $\dot{z}^2 = \dot{x}^2 + \dot{y}^2$ and its Fluxion, and thence obtain Variety of different Forms for each of the quantities CM, ME, EC, as before.

Otherwise

FIG. 48. Otherwise thus, when we have the Perpendicular upon the Tangent drawn to any Point M of the Curve, we can find the Radius of Curvature CM very easily thus; let BP, Bp be perpendicular to the Tangents MP, np, and let $BP=u$, $pq=\acute{u}$, the rest as before. Then by the similar Triangles pMq, MCn; and Mrn, MPB; $r\acute{u}=pM\times Mn=y\acute{y}$, or $r\dot{u}=y\dot{y}$, whence $r=\frac{y\dot{y}}{\dot{u}}$. Hence therefore to find the Radius of Curvature the Rules are,

46. 1. *In Curves whose Ordinates are perpendicular to the Abscissa, for the Curve, Abscissa, and Ordinate, put* z, x, y: *Put the Equation of the Curve (reduced as low and simple as possible) into Fluxions, writing* 1 *for any one of the first Fluxions contained in it; then put the resulting Equation into Fluxions again, writing* ~~one~~ *for the same first Fluxion. From the first Equation the other Fluxion will be determin'd, and by the second you will get the second Fluxion; which being had, substitute them into one of the following Forms that contain this first and second Fluxion.*

But in the vertices of Curves, where they cut the Abscissa at right Angles, take $\frac{y\dot{y}}{\dot{x}}$ *for the Radius of Curvature; and in the highest Point, where they cut the Ordinate at Right Angles,* mH *is the Radius.*

$$\text{If } \dot{x}=1,\left\{\begin{array}{l} \text{Radius } mC=\dfrac{\overline{1+\dot{y}^2}|^{\frac{3}{2}}}{-\ddot{y}}=\dfrac{\dot{z}^2\sqrt{\dot{z}^2-1}}{-\ddot{z}}. \\ mH=\dfrac{1+\dot{y}^2}{-\ddot{y}}=\dfrac{\dot{z}\sqrt{\dot{z}^2-1}}{-\ddot{z}}. \\ CH=\dfrac{\dot{y}\times\overline{1+\dot{y}^2}}{-\ddot{y}}=\dfrac{\dot{z}\times\overline{\dot{z}^2-1}}{-\ddot{z}}. \end{array}\right.$$

FIG.

$$\text{If } \dot{y} = 1, \left\{ \begin{aligned} \text{Radius } mC &= \frac{\overline{1+\dot{x}^2}|^{\frac{3}{2}}}{\ddot{x}} = \frac{\dot{z}^2\sqrt{\dot{z}^2-1}}{\ddot{z}}. \\ mH &= \frac{\dot{x}\times\overline{1+\dot{x}^2}}{\ddot{x}} = \frac{\dot{z}\times\overline{\dot{z}^2-1}}{\ddot{z}}. \\ CH &= \frac{1+\dot{x}^2}{\ddot{x}} = \frac{\dot{z}\sqrt{\dot{z}^2-1}}{\ddot{z}}. \end{aligned} \right.$$

$$\text{If } \dot{z} = 1, \left\{ \begin{aligned} \text{Radius } mC &= \frac{\sqrt{1-\dot{x}^2}}{\ddot{x}} = \frac{\sqrt{1-\dot{y}^2}}{-\ddot{y}}. \\ mH &= \frac{\dot{x}\sqrt{1-\dot{x}^2}}{\ddot{x}} = \frac{1-\dot{y}^2}{-\ddot{y}}. \\ CH &= \frac{1-\dot{x}^2}{\ddot{x}} = \frac{\dot{y}\sqrt{1-\dot{y}^2}}{-\ddot{y}}. \end{aligned} \right.$$

2. *In Curves related to a fixed Point, let the Curve $AM=z$, Ordinate $BM=y$, and $Mr=\dot{x}$. Put the Equation of the Curve into Fluxions, and make some of the first Fluxions invariable, for which write* 1; *and the Equation being turned into Fluxions again, write* 1 *for the invariable Fluxion, as before: Thus you will determine the other Fluxion and second Fluxion, whose Values being substituted into one of the Forms below which contains that first and second Fluxion, you will have the Radius of Curvature.* 47.

But when you know the Perpendicular upon the Tangent drawn to any Point of the Curve, put this perpendicular $=u$; *and find the Value of* $\frac{y\dot{y}}{\dot{u}}$ *from the Equation of the Curve; and this is the Radius of Curvature.* 48.

If $\dot{x}=1$, *Radius* $MC = \frac{y\times\overline{1+\dot{y}^2}|^{\frac{3}{2}}}{1+\dot{y}^2-y\ddot{y}} = \frac{y\dot{z}^2\sqrt{\dot{z}^2-1}}{\dot{z}\sqrt{\dot{z}^2-1}-y\ddot{z}}$.

If $\dot{y}=1$, *Radius* $MC = \frac{y\times\overline{1+\dot{x}^2}|^{\frac{3}{2}}}{\dot{x}+\dot{x}^3+y\ddot{x}} = \frac{y\dot{z}^2\sqrt{\dot{z}^2-1}}{\dot{z}^3-\dot{z}+y\ddot{z}}$.

If $\dot{z}=1$, *Radius* $MC = \frac{y\sqrt{1-\dot{x}^2}}{\dot{x}\sqrt{1-\dot{x}^2}+y\ddot{x}} = \frac{y\sqrt{1-\dot{y}^2}}{1-\dot{y}^2-y\ddot{y}}$.

Also $ME = \frac{MC\times\dot{x}}{\dot{z}}$, *and* $EC = \frac{MC\times\dot{y}}{\dot{z}}$.

Ex.

Ex. 1.

FIG. 49. *Let the Equation of the Curve be* $ax-ay=x^2+y^2$, where $AB=x$, $Bm=y$, $a=$ a given Line. Let $\dot{x}=1$, then putting the Equation into Fluxions $a-a\dot{y}=2x+2y\dot{y}$, and again $-a\ddot{y}=2+2\dot{y}^2+2y\ddot{y}$; whence $\dot{y}=\frac{a-2x}{a+2y}$, and $\ddot{y}=\frac{-2-2\dot{y}^2}{a+2y}=\frac{-2\times\overline{a+2y}^2-2\times\overline{a-2x}^2}{\overline{a+2y}^3}$ $=$(becaufe $ax-ay=x^2+y^2$) $\frac{-4aa}{\overline{a+2y}^3}$. Whence the Radius of Curvature $=\frac{\overline{1+\dot{y}^2}|^{\frac{3}{2}}}{-\ddot{y}}=\frac{\overline{\overline{a-2x}^2+\overline{a+2y}^2}|^{\frac{3}{2}}}{4aa}=\frac{\overline{2a^2}|^{\frac{3}{2}}}{4aa}=\frac{a}{\sqrt{2}}$ a determinate Quantity: Therefore Am is the Arch of a Circle whofe Radius is $\frac{a}{\sqrt{2}}$.

Ex. 2.

50. *Let* Am *be an Ellipfis*, $a=$ Tranfverfe, $b=$ the Parameter, and $abx-bxx=ayy$. In Fluxions $ab-2bx=2ay\dot{y}$ (where $\dot{x}=1$), and this again in Fluxions $-2b=2a\dot{y}^2+2ay\ddot{y}$; hence $\dot{y}=\frac{ab-2bx}{2ay}$, and $-\ddot{y}=\frac{b+a\dot{y}^2}{ay}=\frac{4abyy+\overline{ab-2bx}^2}{4aay^3}=$ (expunging x) $\frac{bb}{4y^3}$. Therefore $mH=\frac{1+\dot{y}^2}{-\ddot{y}}=\frac{4aayy+\overline{ab-2bx}^2}{-\ddot{y}\times 4aayy}=\frac{aabb+4aayy-4abyy}{aabb}y=y+\frac{4a-4b}{abb}y^3$, and Radius $mC=\frac{\overline{1+\dot{y}^2}|^{\frac{3}{2}}}{-\ddot{y}}=\frac{\overline{aabb+4aayy-4abyy}|^{\frac{3}{2}}}{2bba^3}$.

And if Am *were an Hyperbola*, it would be found in the fame Manner that $mH=y+\frac{4a+4b}{abb}y^3$, and $mC=\frac{\overline{a^2b^2+4a^2y^2+4aby^2}|^{\frac{3}{2}}}{2bba^3}$.

Or thus in particular Numbers.

Let $a=3$, $b=1$, and assume $x=1$, $\dot{x}=1$, then $y=\sqrt{bx-\frac{b}{a}xx}=\sqrt{\frac{2}{3}}$, and by the foregoing operation $\dot{y}=\frac{ab-2bx}{2ay}=\frac{1}{6\sqrt{\frac{2}{3}}}$, and $-\ddot{y}=\frac{b+a\dot{y}^2}{ay}$ $=\frac{3\sqrt{3}}{8\sqrt{2}}$. Whence $mC=\frac{\overline{1+\dot{y}^2}|^{\frac{3}{2}}}{-\ddot{y}}=\frac{125}{54}$.

Otherwise thus:

Let Transverse $=2r$, Conjugate $=2c$, B the Focus, $BM=y$, $BT=u$ the Perpendicular on the Tangent at M. By the conic Sections $u=\frac{cy}{\sqrt{2ry-yy}}$, FIG. 57.

and $\dot{u}=\frac{cr y\dot{y}}{\overline{2ry-yy}|^{\frac{3}{2}}}$ whence $\frac{y\dot{y}}{\dot{u}}=\frac{\overline{2ry-yy}|^{\frac{3}{2}}}{cr}$, the Radius of Curvature in M.

Ex. 3.

In the Parobola $ax=yy$, then $a\dot{x}=2y$, and $a\ddot{x}=2$ (putting $\dot{y}=1$); whence Radius $mC=\frac{\overline{1+\dot{x}^2}|^{\frac{3}{2}}}{\ddot{x}}$ 52.

$=\frac{\overline{1+\frac{4yy}{aa}}|^{\frac{3}{2}}}{\frac{2}{a}}=\frac{\overline{aa+4yy}|^{\frac{3}{2}}}{2aa}=\frac{4.\overline{mD}^3}{aa}$.

Otherwise thus:

Let $a=$ Latus rectum, B the Focus, $BM=y$, $BT=u$, the Perpendicular on the Tangent at M. By the Nature of the Figure $uu=\frac{1}{4}ay$, and $\dot{u}=\frac{a\dot{y}}{4\sqrt{ay}}$, 58.

whence $\frac{y\dot{y}}{\dot{u}}=\frac{4y\dot{y}\sqrt{ay}}{a\dot{y}}=\frac{4y\sqrt{ay}}{a}=\frac{8uy}{a}$, the Radius of Curvature in M.

Ex. 4.

FIG. 51. *Let Dm be an Hyperbola between the Assymptotes.* $AB=x$, $Bm=y$. Draw mR parallel to the other Assymptote; then since the Position of the Ordinate mR is given, there is given the Ratio of mR and RB to mB perpendicular to AB: Let $RB=sy$, $mR=ty$, $AR=x-sy$. By the Nature of the Figure $AR\times Rm = aa = txy - tsyy$, which turned into Fluxions $t\dot{x}+ty\dot{x}+2tsy=0$, and again $2t\dot{x}+ty\ddot{x}-2ts=0$; hence $\dot{x}=\frac{2sy-x}{y}$, and $\ddot{x}=\frac{2s-2\dot{x}}{y}=\frac{2x-2sy}{yy}$. Then $\frac{\overline{1+\dot{x}^2}|^{\frac{3}{2}}}{\ddot{x}}=\frac{\overline{yy+\overline{2sy-x}|^2}|^{\frac{3}{2}}}{2xy-2syy}$, the Radius of Curvature in m.

And in a right-angled Hyperbola, $s=0$, and the Radius becomes $\frac{\overline{yy+xx}|^{\frac{3}{2}}}{2xy}=\frac{\overline{yy+xx}|^{\frac{3}{2}}}{2aa}=\frac{Am^3}{2aa}$.

Ex. 5.

52. *Let Am be the Cissoid*, whose Equation is $axx-yx^2=y^3$, then $a-y=\frac{y^3}{xx}$, in Fluxions $-1=\frac{3yy}{xx}-\frac{2y^3\dot{x}}{x^3}$, reduced $-x^3=3y^2x-2y^3\dot{x}$, in Fluxions $-3x^2\dot{x}=6yx+3y^2\dot{x}-6y^2\dot{x}-2y^3\ddot{x}$; hence $\dot{x}=\frac{3xy^2+x^3}{2y^3}$, and $\ddot{x}=\frac{6xy+\overline{3xx-3yy}\times\dot{x}}{2y^3}=\frac{3xy^4+6x^3y^2+3x^5}{4y^6}$. Then the Radius $=\frac{\overline{1+\dot{x}^2}|^{\frac{3}{2}}}{\ddot{x}}=\frac{\overline{4y^6+9x^2y^4+6x^4y^2+x^6}|^{\frac{3}{2}}}{2y^3\times:3xy^4+6x^3y^2+3x^5}$ $=$ (expunging y^3, y^6) $\frac{\overline{4aa-8ay+10yy+xx+9ay-9yy}|^{\frac{3}{2}}\times xx}{6aay}$ $=\frac{\overline{4aa+ay+y^2+x^2}|^{\frac{3}{2}}\times x}{6aay}=$ (expunging xx) $\frac{\overline{4a-3y}|^{\frac{3}{2}}\times a\sqrt{y}}{6\times\overline{a-y}^2}$. Therefore in the Vertex A, the Radius of Curvature is 0.

Or

Or thus definitely in Numbers.

Let $a=10$, $\dot{y}=1$, and assume $y=2$, then $x=1$; and $\dot{x} = \frac{3xy^2+x^3}{2y^3} = \frac{13}{16}$. And $\ddot{x} = \frac{6xy+\overline{3x^2-3yy}\times\dot{x}}{2y^3} = \frac{75}{16\times 16}$; whence $\frac{\overline{1+\dot{x}^2}|^{\frac{3}{2}}}{\ddot{x}} = \frac{85}{48}\sqrt{17}$.

Ex. 6.

Let $x=y^n$ *be an equation for all Parabolas,* then (if $\dot{y}=1$) $\dot{x}=ny^{n-1}$, $\ddot{x}=\overline{nn-n}\times y^{n-2}$. Therefore $\frac{\overline{1+\dot{x}^2}|^{\frac{3}{2}}}{\ddot{x}} = \frac{\overline{1+nny^{2n-2}}|^{\frac{3}{2}}}{n.\overline{n-1}.y^{n-2}}$.

Hence it will easily appear that every Parabola, except the Appolonian, has the Radius of Curvature in the Vertex either infinite or nothing at all.

Cor. If $ax=yy$, $bx=y^3$, $cx=y^4$, $dx=y^5$, &c. be a Series of Parabolas, then the Angle of Contact (made with the Tangent and Curve) at the Vertex of any one is infinitely greater than the Angle of Contact of the next following one. The Angle of Contact of the first $ax=y^2$, is of the same kind with that of Circles.

Now it appears that the Angles of Contact of one kind infinitely exceed those of another kind, since the Radius of Curvature of one kind is infinitely greater than that of another; and the Curvature or Angle of Contact is reciprocally as that Radius, Thus in the Curve $cx=y^4$, the Radius $=\frac{c}{12yy}=$ Infinity, and in the Curve $dx=y^5$, the Radius $=\frac{d}{20y^3}=$ Infinity, because $y=0$; and the latter Radius is infinitely greater than the former, for $\frac{d}{20y^3}=\frac{c}{12yy}\times\frac{12d}{20cy}$; and therefore the Angle of Contact of the first infinitely exceeds the Angle of Contact of the last. Therefore a Curve of one kind, how great soever it may

FIG. be, cannot be interposed at the Point of Contact between the Curve of another kind and its Tangent, however small that Curve may be: Or an Angle of Contact of one kind cannot necessarily contain an Angle of Contact of another kind, as a whole contains a part.

Likewise in this Series of Parabolic Curves $ax = y^{\frac{1}{2}}$, $bx = y^{\frac{2}{3}}$, $cx = y^{\frac{3}{4}}$, $dx = y^{\frac{4}{5}}$, *&c*; the Angle of Contact which any Curve makes with the Absciss at the Vertex is infinitely greater than the next preceding. The Angle of Contact of the first $ax = y^{\frac{1}{2}}$ is of the same kind with Circles. And though the Angles of the succeeding Curves do infinitely exceed the preceding ones, yet they can never arrive at the magnitude of right lined Angles. Moreover between the Angles of Contact of any two of these may other Angles of Contact be found *ad infinitum*, that will infinitely exceed each other.

Ex. 7.

54. *Let Dm be the logarithmic Curve, whose Equation is* $y\dot{x} = a\dot{y}$, putting $AB = x$, $Bm = y$, $a =$ Subtangent. Then (if $\dot{x} = 1$) $\dot{y} = \frac{y}{a}$, and $\ddot{y} = \frac{\dot{y}}{a} = \frac{y}{aa}$; therefore $\frac{\overline{1+\dot{y}^2}|^{\frac{3}{2}}}{-\ddot{y}} = \frac{\overline{aa+yy}|^{\frac{3}{2}}}{-ay}$, the Radius of Curvature; where the negative Sign only shews its Position.

Ex. 8.

52. *Let Am be the Catenary*, $AB = x$, $Bm = y$, $Am = z$; It's Equation is $zz = 2ax + xx$, then (putting $\dot{z} = 1$) it's Fluxion is $z = a\dot{x} + x\dot{x}$, and again $1 = a\ddot{x} + x\ddot{x} + \dot{x}^2$; hence $\dot{x} = \frac{z}{a+x}$, and $\ddot{x} = \frac{1-\dot{x}^2}{a+x} = \frac{\overline{a+x}^2 - zz}{\overline{a+x}^3}$ $= \frac{aa}{\overline{a+x}^3}$. Therefore $\frac{\sqrt{1-\dot{x}^2}}{\ddot{x}} = \frac{\overline{a+x}^3\sqrt{\overline{a+x}^2 - zz}}{\overline{a+x} \times aa}$ $= \frac{\overline{a+x}^2}{aa}\sqrt{aa} = \frac{\overline{a+x}^2}{a} = a + \frac{zz}{a}$.

Ex. 9.

Ex. 9.

Let Am be the Cycloid, $AB=x$, $Bm=y$, $AD=v$, $AR=a$, then $BD=\sqrt{ax-xx}$, and $\dot{v}=\frac{a}{2\sqrt{ax-xx}}$, by the Nature of the Circle (putting $\dot{x}=1$). By the Property of the Cycloid, $y=v+\sqrt{ax-xx}$, whence $\dot{y}=\dot{v}+\frac{a-2x}{2\sqrt{ax-xx}}=\frac{a-x}{\sqrt{ax-xx}}=\sqrt{\frac{a-x}{x}}$; and $\ddot{y}=\frac{-a}{2x^{\frac{3}{2}}\sqrt{a-x}}$; whence the Radius $=\frac{\overline{1+\dot{x}^2}|^{\frac{3}{2}}}{-\ddot{y}}=2\sqrt{aa-ax}=2DR$. FIG. 55.

Ex. 10.

Let Dm be the Conchoid, $RA=b$, $AD=c$, $AB=x$, $Bm=y$; it's Equation is $\overline{b+y}\sqrt{cc-yy}=xy$. This squared and divided by yy, $\frac{bbcc}{yy}+\frac{2bcc}{y}+cc-bb-2by-yy=xx$. In Fluxions, $-\frac{2bbcc}{y^3}-\frac{2bcc}{yy}-2b-2y=2x\dot{x}$; and again $\frac{6bbcc}{y^4}+\frac{4bcc}{y^3}-2=2\dot{x}^2+2x\ddot{x}$; whence $\dot{x}=\frac{-bbcc}{y^3x}-\frac{bcc}{yyx}-\frac{b}{x}-\frac{y}{x}$, and $\ddot{x}=\frac{3bbcc}{y^4x}+\frac{2bcc}{y^3x}-\frac{1}{x}-\frac{\dot{x}^2}{x}$. Now to find the Radius of Curvature in any given Point m; let $b=5$, $c=10$, if you take $y=6$, then $x=14\frac{2}{3}$, and $\dot{x}=-2\frac{35}{72}$, and $\ddot{x}=\frac{409}{1056}$; whence $\frac{\overline{1+\dot{x}^2}|^{\frac{3}{2}}}{\ddot{x}}=40,326$ the Radius of Curvature in that Point m. 53.

The Radius of Curvature may alſo be expreſſed indefinitely in the ſame Manner as in the foregoing, though more prolixly. Let $\dot{x}=\frac{p}{x}$, $\ddot{x}=\frac{q}{x}-\frac{\dot{x}^2}{x}$

FIG. $\frac{\dot{x}^2}{\ddot{x}} = \frac{qxx - pp}{xx\ddot{x}}$, then $\overline{1+\dot{x}^2}|^{\frac{3}{2}} = \frac{\overline{xx+pp}|^{\frac{3}{2}}}{x^3}$: Whence $\frac{\overline{1+\dot{x}^2}|^{\frac{3}{2}}}{\ddot{x}} = \frac{\overline{xx+pp}|^{\frac{3}{2}}}{qxx - pp}$; therefore either x or y being given, the other will be known, and thence the quantities p, q. For Example in the Vertex, where $x=0$, the Radius becomes $\frac{p^3}{-pp} = -p = \frac{bbcc}{y^3} + \frac{bcc}{yy} + b + y = \frac{bb}{c} + 2b + c = \frac{\overline{b+c}^2}{c}$.

Ex. 11.

56. *Let Dm be the Quadratrix*, $AB=x$, $Bm=y$, AF or $AR=r$, $AE=u$, $CE=v$, Arch $CF=t$, $RCF=q$, $RC=q-t$, $Am=s$. By the Nature of the Circle, the Fluxion of RC $(-\dot{t})$ is to the Fluxion of AE $(\dot{u})$ as r to $\dot{v}$, that is $-v\dot{t} = r\dot{u}$, likewise $r\dot{v} = u\dot{t}$, and by the Property of the Figure $t=y$, and $\dot{t} = \dot{y}$; and by similar Triangles $uy=vx$, in Fluxions $y\dot{u} + u\dot{y} = x\dot{v} + v\dot{x}$, and expunging $\dot{v}$, $\dot{u}$, v, we get $-y^2\dot{y} + rx\dot{y} - x^2\dot{y} = ry\dot{x} = -y^2 + rx - xx$ (putting $\dot{y}=1$), this in Fluxions $ry\ddot{x} + r\dot{x} = -2y + r\dot{x} - 2x\dot{x}$. Therefore $\dot{x} = \frac{-y^2 + rx - xx}{ry} = \frac{rx - ss}{ry}$; and $\ddot{x} = \frac{-2y - 2x\dot{x}}{ry} = \frac{2ssx - 2ssr}{rryy}$, hence $\frac{\overline{1+\dot{x}^2}|^{\frac{3}{2}}}{\ddot{x}} =$

$$\frac{\overline{rryy + \overline{rx - ss}^2}|^{\frac{3}{2}} \times rryy}{r^3y^3 \times 2ssx - 2ssr} = \frac{-s}{2ry} \times \frac{\overline{rr - 2rx + ss}|^{\frac{3}{2}}}{r - x}.$$

If the Radius of Curvature is required in the Vertex F, it will be found by expressing AE and CE by infinite Series in Terms of y, rejecting the superfluous Terms. Thus it will hereafter be shewn (Ex. 5. Prob. VIII.) that $CF = CE + \frac{CE^3}{6rr}$ &c; that is $y = v + \frac{v^3}{6rr}$, whence by Reversion of Series $v=y$

$-\frac{y^3}{6rr}$, and thence $u=\sqrt{rr-vv}=r-\frac{yy}{2r}$, whence FIG.
$uy=vx$ becomes $ry-\frac{y^3}{2r}=yx-\frac{y^3}{6rr}$, and $x=r-\frac{yy}{3r}$, and $\dot{x}=-\frac{2y\dot{y}}{3r}$. But the Radius of Curvature in F is $\frac{y\dot{y}}{\dot{x}}$, where you must write $r-x$ for x, $-\dot{x}$ for $\dot{x}$. Then $\frac{y\dot{y}}{\dot{x}}=\frac{y\dot{y}}{2y\dot{y}}\times 3r=\frac{3}{2}r$.

Ex. 12.

Let AM be the Logarithmetic Spiral; TM a Tangent; BT, BP perpendicular to BM, TM. Let $BM=y$, $BP=u$, the given Ratio of MT to BT as c to t; then $u=\frac{ty}{c}$; and $\dot{u}=\frac{t\dot{y}}{c}$, whence $MC=\frac{y\dot{y}}{\dot{u}}=\frac{cy}{t}$; therefore C falls in the Line TB produced. 59.

Ex. 13.

Let BM be Archimedes's Spiral, AB=a, Arch $AD=v$, $BM=y$, and let $by=av$, whence $b\dot{y}=a\dot{v}$; but by similar Sectors $\dot{v}=\frac{a\dot{x}}{y}$ or $\dot{v}=\frac{a\dot{x}}{y}$, therefore $aa\dot{x}=by\dot{y}$. Let $\dot{y}=1$, then $\dot{x}=\frac{by}{aa}$, and $\ddot{x}=\frac{b\dot{y}}{aa}$; whence the Radius $=\frac{y\times\overline{1+\dot{x}^2}|^{\frac{3}{2}}}{\dot{x}+\dot{x}^3+y\ddot{x}}=\frac{\overline{a^4+bbyy}|^{\frac{3}{2}}}{b^3y^2+2ba^4}=\frac{a\times\overline{a^2+vv}|^{\frac{3}{2}}}{bv^2+2ba^2}$. 60.

Ex. 14.

Suppose the Equation of the Spiral to be $by^m=a^mv$. Then, if $\dot{x}=1$, $mby^{m-1}\dot{y}=a^m\dot{v}=\frac{a^{m+1}\dot{x}}{y}$, or $mby^m\dot{y}=a^{m+1}$, again in Fluxions $m^2by^{m-1}\dot{y}^2+mby^m\ddot{y}=0$; hence 60.

FIG. hence $\dot{y}=\frac{a^{m+1}}{mby^{m}}$, and $\ddot{y}=\frac{-a^{2m+2}}{mb^{2}y^{2m+1}}$. Whence the Radius $=\frac{y\times\overline{1+\dot{y}^{2}}|^{\frac{3}{2}}}{1+\dot{y}^{2}-y\ddot{y}}=\frac{\overline{m^{2}b^{2}y^{2m}+a^{2m+2}}|^{\frac{3}{2}}}{m^{3}b^{3}y^{3m-1}+\overline{m+1}.mby^{m-1}a^{2m+2}}$.

Ex. 15.

61. *Let BM be the reciprocal or hyperbolic Spiral,* Radius *AB* or *BD*$=a$, given Arch *AE*$=n$, Arch *AF* $=v$, *BM*$=y$. By the Property of the Curve $an=vy$, whence $y\dot{v}+v\dot{y}=0$, and $\dot{v}=\frac{-v\dot{y}}{y}=\frac{-an\dot{y}}{yy}$: But by similar Triangles $\dot{v}=\frac{a\dot{x}}{y}$, therefore $y\dot{x}=-n\dot{y}$, let $\dot{x}=1$, then $\dot{y}=\frac{-y}{n}$, and $\ddot{y}=\frac{-\dot{y}}{n}=\frac{y}{nn}$. Therefore $\frac{y\times\overline{1+\dot{y}^{2}}|^{\frac{3}{2}}}{1+\dot{y}^{2}-y\ddot{y}}=\frac{y\times\overline{n^{2}+y^{2}}|^{\frac{3}{2}}}{n^{3}}=$ (because the Subtangent *BT*$=n$) $\frac{y\times TM^{3}}{n^{3}}=\frac{y\times GM^{3}}{y^{3}}=\frac{GM^{3}}{y^{2}}$, supposing *BG* perpendicular to *BM*.

Ex. 16.

62. *Let BM be a kind of Spiral, Radius BD*$=a$, *Arch DF*$=t$, *BM*$=y$, *Arch MR*$=v$, *and let its Nature be* $a^{2}y=v^{3}$. By the similar Sectors *BMR* and *BFD*, $v=\frac{ty}{a}$, whence $a^{5}=t^{3}y^{2}$, in Fluxions $3y^{2}t^{2}\dot{t}+2t^{3}y\dot{y}=0$, by similar fluxionary Triangles $\dot{t}=\frac{a\dot{x}}{y}$, and expunging $\dot{t}$, $3a\dot{x}+2t\dot{y}=0$, let $\dot{y}=1$, then $\dot{x}=\frac{-2t}{3a}$, and $\ddot{x}=\frac{-2\dot{t}}{3a}=\frac{-2\dot{x}}{3y}=\frac{4t}{9ay}=\frac{4v}{9yy}$: Therefore $\frac{y\times\overline{1+\dot{x}^{2}}|^{\frac{3}{2}}}{\dot{x}+\dot{x}^{3}+y\ddot{x}}=\frac{y\times\overline{9yy+4vv}|^{\frac{3}{2}}}{-v\times 6yy+8vv}=\frac{-v^{3}}{a^{4}}\times\frac{\overline{9v^{4}+4a^{4}}|^{\frac{3}{2}}}{6v^{4}+8a^{4}}$.

Or

Or thus, in particular Numbers. FIG.

Let $y=1$, $v=4$, $\dot{y}=1$, then $\dot{x}=\frac{-2t}{3a}=\frac{-2v}{3y}$ $=\frac{-8}{3}$, and $\ddot{x}=\frac{-2\dot{x}}{3y}=\frac{16}{9}$. Whence $\frac{y\times\overline{1+\dot{x}^2}|^{\frac{3}{2}}}{\dot{x}+\dot{x}^3+y\ddot{x}}$ $=\frac{-73\sqrt{73}}{536}$ the Radius of Curvature, where the negative Sign relates only to the Position of it.

SCHOLIUM.

Let there be any geometrical Curve defined by this general Equation $e+fx^m+gy^n+hx^ry^s=0$; make $AB=x$, $BM=y$, the Perpendicular to the Curve $MQ=\pi$: Then the Radius of Curvature may be expressed in algebraic Terms affected with π, thus; compute the value of $\ddot{y}$ in the foregoing general Equation (putting $\dot{x}$ invariable), which substitute in the Quantity $\frac{\dot{z}^3}{-\dot{x}\ddot{y}}$ and put $\frac{\pi}{y}$ for $\frac{\dot{z}}{\dot{x}}$; and the Radius of Curvature will be $=$ 63.

$$\frac{\overline{ngy^{n-1}+shx^ry^{s-1}}|^3\times\pi^3}{\left.\begin{array}{l}+\overline{m.\overline{m-1}.fx^{m-2}+r.\overline{r-1}.hx^{r-2}y^s}\times\overline{ngy^{n-1}+shx^ry^{s-1}}|^2\\-2rshx^{r-1}y^{s-1}\times\overline{mfx^{m-1}+rhx^{r-1}y^s}\times\overline{ngy^{n-1}+shx^ry^{s-1}}\\+\overline{n.\overline{n-1}.gy^{n-2}+s.\overline{s-1}.hx^ry^{s-2}}\times\overline{mfx^{m-1}+rhx^{r-1}y^s}|^2\end{array}\right\}\times y^3}.$$

Example.

Let $bx^2-ay^2+abx=0$, *represent an Hyperbola*, $a=$ *Transverse*, $b=$ *Parameter.* Comparing this with the general Form, we shall find $e=0$, $f=b$, $m=2$, $g=-a$, $n=2$, $h=ab$, $r=1$, $s=0$, whence the Radius of Curvature $=\frac{\overline{-2ay}|^3\times\pi^3}{2bx\times\overline{-2ay}|^2-2a\times\overline{2bx+ab}|^2}$ $=$ (expunging x) $\frac{\overline{-2ay}|^3\,\pi^3}{-2a^3b^2y^3}=\frac{4\pi^3}{bb}$. And it will be the same for the Ellipsis.

 PROB.

PROB. VI.

To determine the (Quality or Degree of) Variation of Curvature in any Point of a Curve.

FIG. The Variation of Curvature depends on the Moment of the Radius of Curvature, and the Moment of the Curve; and it is as the Moment of the Radius of Curvature if the Moment of the Curve is given; and reciprocally as the Moment of the Curve if the Moment of the Radius is given. Therefore this Variation is as the Fluxion of the Radius of Curvature divided by the Fluxion of the Curve: Therefore,

1. *Find the Radius of Curvature by the last Problem, which divide by the Fluxion of the Curve; and the Fluxions may be exterminated by the Equation of the given Curve, or perhaps by expressing their Ratio by Help of the Tangent, Ordinate, or Subnormal.*

2. In Curves referred to an Axis, finding the Fluxion of the Quantity $\frac{\dot{z}^3}{-\ddot{y}}$ which divide by $\dot{z}$, putting $\dot{x}=1$, and exterminating $\dot{z}$, $\ddot{z}$, you will get the following Rule. *Put x for the Abscissa, y for the Ordinate, $\dot{x}=1$: Then substitute the Values of $\dot{y}$, $\ddot{y}$, $\dddot{y}$ (got from the Equation of the Curve) into the Quantity* $\frac{-3\dot{y}\ddot{y}^2+\dddot{y}\times\overline{1+\dot{y}^2}}{\ddot{y}^2}$, *and it will give the Variation.*

Example 1.

52. *Let the Curve be a Parabola*, then $ax=yy$, and $\dot{y}=\frac{a}{2y}$, $\ddot{y}=\frac{-a\dot{y}}{2yy}$, $\dddot{y}=\frac{2a\dot{y}^2-ay\ddot{y}}{2y^3}$. Now if $a=2$, $x=\frac{1}{2}$, $y=1$, then $\dot{y}=1$, $\ddot{y}=-1$, $\dddot{y}=3$, whence $\frac{-3\dot{y}\ddot{y}^2+\dddot{y}\times\overline{1+\dot{y}^2}}{\ddot{y}^2}=3$. Likewise if $x=2$, $y=2$, then the Variation $=6$.

In

In general,

Since $ax = y^2$, therefore $\dot{y} = \frac{a}{2y}$, and $\ddot{y} = \frac{-a\dot{y}}{2yy}$ FIG.
$= \frac{-aa}{4y^3}$, and $\dot{\ddot{y}} = \frac{2a\dot{y}^2 - ay\ddot{y}}{2y^3} = -\frac{3aa\dot{y}}{4y^4} = \frac{3a^3}{8y^5}$.
Whence $\frac{-3\dot{y}\ddot{y}^2 + \dot{\ddot{y}} \times \overline{1+\dot{y}^2}}{\ddot{y}^2} = \frac{-3a^5}{32y^7} + \frac{3a^3}{8y^5}$
$+ \frac{3a^5}{32y^7} : \times \frac{16y^6}{a^4} = \frac{6y}{a}$.

Ex. 2.

In the Ellipsis let $bx - \frac{b}{a}xx = yy$. Suppose $a = \frac{2}{3}$, 50.
$b = 2$, to find the Variation of Curvature when $x = \frac{1}{2}$, and $y = \frac{1}{2}$. By the Process of Example 2 of the last Problem, $\dot{y} = \frac{ab - 2bx}{2ay} = -1$, $\ddot{y} = \frac{b + a\dot{y}^2}{-ay} = -8$, and it will be found that $\dot{\ddot{y}} = \frac{b + a\dot{y}^2 - 2ay\ddot{y}}{ayy} \times \dot{y} = -48$. Whence $\frac{-3\dot{y}\ddot{y}^2 + \dot{\ddot{y}} \times \overline{1+\dot{y}^2}}{\ddot{y}^2} = \frac{3}{2}$ the Variation required.

Or in general thus:

By the last Problem the Radius of Curvature is $\frac{\overline{aabb + 4aayy - 4abyy}|^{\frac{3}{2}}}{2bba^3}$, and its Fluxion divided by $\dot{z}$ is $\frac{a-b}{bbaa\dot{z}} \times 6y\dot{y}\sqrt{aabb + 4aayy - 4abyy}$. But from the Equation of the Curve $\frac{\dot{y}}{\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}} = \frac{ab - 2bx}{\sqrt{aabb + 4aayy - 4abyy}}$; whence the Variation of Curvature becomes $\frac{a-b}{baa} \times \overline{\tfrac{1}{2}a - x} \times 12y$; and is therefore every where as the Rectangle mBO, supposing O the Center.

Ex. 3.

FIG. 52. *Let* $x=y^3$ *denote a cubic Parabola*; then $3y^2\dot{y}=1$, and $\dot{y}=\frac{1}{3yy}$, $\ddot{y}=\frac{-2\dot{y}}{3y^3}=\frac{-2}{9y^5}$, $\dot{\ddot{y}}=\frac{10\dot{y}}{9y^6}=\frac{10}{27y^8}$; whence $-3\dot{y}+\frac{\dot{\ddot{y}}}{\ddot{y}^2}\times\overline{1+\dot{y}^2}=\frac{15y^2}{2}-\frac{1}{6yy}$ the Variation required.

Ex. 4.

55. *In the Cycloid Am* the Radius of Curvature is $2\sqrt{aa-ax}$ and its Fluxion $=\frac{-a\dot{x}}{\sqrt{aa-ax}}$; but by the Property of the Figure $\dot{z}=\frac{AD\times\dot{x}}{x}=\frac{DR\times\dot{x}}{y}=\frac{\dot{x}\sqrt{aa-ax}}{y}$, whence $\frac{-a\dot{x}}{\dot{z}\sqrt{aa-ax}}=\frac{-ay}{aa-ax}=\frac{-y}{BR}=-\frac{AB}{BD}$. The negative Sign only ſhews that the Curvature increaſes.

Ex. 5.

59. *In the Logarithmic Spiral*, (by Ex. 12. Prob. V.) the Radius of Curvature is $\frac{cy}{t}$ (putting $c:t:d::$ $TM:TB:BM$), and its Fluxion is $\frac{c\dot{y}}{t}$, and $\dot{z}=\frac{c\dot{y}}{d}$; therefore $\frac{c\dot{y}}{t\dot{z}}=\frac{d}{t}$ for the Variation of Curvature, which therefore is uniform.

Ex. 6.

61. *Let BM be the hyperbolic Spiral*, (Ex. 15. Prob. laſt) the Radius of Curvature is $\frac{y\times\overline{n^2+y^2}|^{\frac{3}{2}}}{n^3}$, whoſe Fluxion is $\frac{n^2+4y^2}{n^3}\dot{y}\sqrt{n^2+y^2}$, but the Subtangent $BT=n$, therefore $\dot{z}=\frac{\dot{y}}{y}\sqrt{nn+yy}$, by which divide the other Quantity, then $\frac{n^2y+4y^3}{n^3}=$ Variation of Curvature in *M*.

PROB.

PROB. VII.

To find the Nature of the Curve, by whose Evolution a given Curve is described.

A Curve ADM is said to be described by the Evolution of another Curve VEC, when a Line is supposed to be applied to the convex Side of the Curve VEC, and the End of it A is made to move with a circular Motion, and to describe the Curve ADM; whilst the Part DE or MC that continually leaves the Curve VEC is extended into a right Line. The Curve VEC is called the Evolute of ADM. FIG. 64.

Hence it appears, that the Line CM which is a Tangent at C to the Curve VEC, is the Radius of Curvature in M of the Curve ADM, and is equal in Length to the Curve CE + the right Line ED the Tangent at E; or to the Length of the Curve CEV + the Line VA. Now the Evolute VEC being the Locus of the Center of Curvature of all the Points of the Curve ADM, the Nature of the Curve VEC is to be found. Therefore,

1. *In Curves related to an Axis, put the Abscissa* $AB = x$, *perpendicular Ordinate* $BM = y$, *and let* CM *be the Radius of Curvature, and draw* CH *parallel to* AB, *and* CL *to* BM, *and produce* MB *to* H, *and put* v, u, *for the Abscissa and Ordinate of the Curve* VC *referred to the Axis* AL; *then find* MH, HC *by Prob. V. and expunging* x *and* y, *the Nature of the Curve* VC *will be discovered.*

2. *And in Spirals find the Radius of Curvature* MC *by Prob. V. which will give all the Points* C *in the Evolute.*

Corol. Hence it will be easy to find any Number of Curves whose Lengths may be expressed by finite Equa-

FIG. Equations; by assuming any Curve AM at Pleasure, and finding its Evolute EC: For the Radius of Curvature MC will give the Length of the Evolute; or $MC - DE =$ Length of the Part EC of the Curve.

Ex. 1.

64. *Let AM be a Parabola*, $ax = yy$, and putting $\dot{y} = 1$, we shall find (by Prob. V.) $CH = \frac{1 + \dot{x}^2}{\ddot{x}} = \frac{1}{2}a + 2x$, and $MH = \dot{x} \times \frac{1 + \dot{x}^2}{\ddot{x}} = CH \times \dot{x} = y + \frac{4xy}{a} = y + \frac{4y^3}{aa}$, and when x and y is o, $AV = \frac{1}{2}a$. Put $v = VL$, $u = LC$, and we have $v = (AB - AV + BL = x - \frac{1}{2}a + \frac{1}{2}a + 2x =)\ 3x$, and $u = (MH - MB =) \frac{4y^3}{aa} = \frac{4x\sqrt{ax}}{a}$: Whence $x^3 = \frac{v^3}{27} = \frac{auu}{16}$; or $\frac{16v^3}{27a} = uu$, for the Property of the Curve VC, which therefore is a semi-cubical Parabola whose Latus rectum is $\frac{27}{16}a$.

Ex. 2.

69. *Let GM be the logarithmic Curve*, $AG = 1$, $AB = x$, $BM = y$. Subtangent $RB = a$, then $a\dot{y} = y\dot{x}$, and (by Prob. V.) $MH = \frac{-aa - yy}{y}$, and $HC = \frac{-aa - yy}{a}$. Let $v = AD = x - \frac{aa + yy}{a}$, $u = DC = 2y + \frac{aa}{y}$, whence $\dot{v} = \dot{x} - \frac{2y\dot{y}}{a} = \frac{aa - 2yy}{ay}\dot{y}$; and $\dot{u} = 2\dot{y} - \frac{aa\dot{y}}{yy} = \frac{2yy - aa}{yy}\dot{y}$; therefore $\dot{y} = \frac{ay\dot{v}}{aa - 2yy} = \frac{-yy\dot{u}}{aa - 2yy}$, or $a\dot{v} = -y\dot{u}$, the Equation of the Curve wherein the Center C is always found.

Now since $\dot{v} = \frac{aa - 2yy}{ay}\dot{y}$, therefore v will increase when aa is greater than $2yy$, and decrease when $2yy$ exceeds aa. Likewise u will increase when $2yy$ exceeds

exceeds aa, that is when v decreases; and it will decrease when v increases.

Ex. 3.

Let AM be the Cycloid, $AB=x$, $BM=y$, Arch $AD=s$, Axis $TR=c$, $AR=a$; then $y=s+\sqrt{ax-xx}$, $\dot{y}=\sqrt{\frac{a-x}{x}}$, $\ddot{y}=\frac{-a}{2x\sqrt{ax-xx}}$; whence $MH=\frac{1+\dot{y}^2}{-\ddot{y}}=2\sqrt{ax-xx}$, $CH=\dot{y}\times MH=2a-2x$. Let $RN=v$, $NC=u$; then $v=a-x$, and $u=y-2\sqrt{ax-xx}=s-\sqrt{ax-xx}$, that is (expunging x) $u=s-\sqrt{av-vv}$: And when $x=0$, $RP=a$. Hence the Curve TCP is also a Cycloid equal to the Cycloid AMT. For compleating the Parallelogram $RTQP$, and describing the semicircle TSQ, $NP=(a-v=)\ AB$, $ES=(\sqrt{av-vv}=)\ BD$, Arch $AD=QS$, and $EC=(c-u=c-s+\sqrt{av-vv}=EN-QS+ES=)\ TS+ES$: Therefore TC is the Cycloid. FIG. 65.

Ex. 4.

Suppose AM to be the Catenary, $AB=x$, $BM=y$, $AM=z$, and $z=\sqrt{2ax+xx}$, and $\dot{x}=\frac{z}{a+x}$, $\ddot{x}=\frac{aa}{\overline{a+x}^3}$ (putting $\dot{z}=1$) whence $MH=\frac{\dot{x}\sqrt{1-\dot{x}^2}}{\ddot{x}}=\frac{z\times\overline{a+x}}{a}$; and $CH=\frac{1-\dot{x}^2}{\ddot{x}}=a+x$, and when $x=0$, HC or $AV=a$. Let $VL=v$, $LC=u$, and we have $v=2x$, and $u=\frac{z\times\overline{a+x}}{a}-y=\frac{a+x}{a}\times\sqrt{2ax+xx}-y$, whence $u=\frac{2a+v}{2a}\sqrt{av+\frac{1}{4}vv}-2.30258\,a\times$ Logarithm of $\frac{2a+v+\sqrt{av+\frac{1}{4}vv}}{2a}$, because $y=2.3025\,a\times$ into that Log. 64.

Ex. 5.

Ex. 5.

FIG. 66. *Suppose VM to be a Curve whose Tangent MT is = the given Line a, AB=x, BM=y, VM=z,* $\dot{y}=1$.

Then by the Nature of the Curve $\dot{y}=\frac{-y\dot{x}}{\sqrt{aa-yy}}=\frac{-y\dot{z}}{a}$; hence $\dot{z}=\frac{-a}{y}$, $\ddot{z}=\frac{a}{yy}$. Then $MH=\frac{aa-yy}{-y}$, $HC=-\sqrt{aa-yy}$, the negative Signs shew they must be taken upward (and to the right Hand): But in the Vertex V, MH and HC are nothing, therefore the Evolution begins in V.

Let $AL=v$, $LC=u$, then $v=\frac{aa-yy}{y}+y-a=\frac{aa-ay}{y}$, and $y=\frac{aa}{a+v}$: Also $u=x+\sqrt{aa-yy}$, and $\dot{u}=\dot{x}-\frac{y\dot{y}}{\sqrt{aa-yy}}=\frac{-\dot{y}}{y}\sqrt{aa-yy}-\frac{y\dot{y}}{\sqrt{aa-yy}}=\frac{-aa\dot{y}}{y\sqrt{aa-yy}}=$ (expunging y and $\dot{y}$) $\frac{a\dot{v}}{\sqrt{2av+vv}}$, an Equation to the Catenary, therefore VC is the Catenary Curve.

Ex. 6.

70. *Let AM be the Cissoid whose Equation is* $ax^2-yx^2=y^3$, and suppose $a=10$. To find the Point C of the Evolute when $y=2$, $x=1$. By a former Calculation (Example 5. Prob. V.) $\dot{x}=\frac{3xyy+x^3}{2y^3}=\frac{13}{16}$, $\ddot{x}=\frac{6xy+\overline{3xx-3yy}\times\dot{x}}{2y^3}=\frac{75}{16\times16}$; whence (by Prob. V.) $HC=\frac{1+\dot{x}^2}{\ddot{x}}=5\frac{2}{3}=BL$, and $MH=\dot{x}\times HC=4\frac{29}{48}$, and $LC=2\frac{29}{48}$. And so for any other given Point of the Curve.

Ex. 7.

Ex. 7.

Let BM be the Log. Spiral, TBC perpendicular to *BM*; and let $c : t : s :: TM : TB : BM$. Then (by Ex. 12. Pr. V.) $MC = \frac{cy}{t}$, and $BC = \frac{sy}{t}$. Therefore since the Angle *BCM* is equal *BMT*, the Curve *BC* is also a Log. Spiral, the same with *BM*. FIG. 67.

Ex. 8.

Suppose BM an hyperbolic Spiral, BE perpendicular to *BM*, *MC* perpendicular to the Tangent in *M*; Let Radius $BA = a$, Arch $AO = n$, $BM = y$, then (by Ex. 15. Pr. V.) Radius of Curvature in $M = \frac{EM^3}{yy}$; therefore draw *EG* perpendicular to *EM* and *GC* perpendicular to *GM*, and the Point *C* is in the Evolute required. 68.

PROB. VIII.

To determine the Length of Curve Lines.

Draw the Ordinate *BM* perpendicular to the Axis *AB*, and *nb* parallel and infinitely near to *MB*. And in Spirals draw *Bn* infinitely near *BM*; and let *Mr* be perpendicular to *nB* or *nb*. Let the Abscissa $AB = x$, Ordinate $BM = y$, Curve $AM = z$, $Mr = \acute{x}$, $rn = \acute{y}$, $Mn = \acute{z}$. In the right-angled Triangle *Mnr*, $\acute{z} = \sqrt{\acute{x}^2 + \acute{y}^2}$, therefore $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$; wherefore, 71. 72.

By the Nature of the Curve exterminate $\dot{x}$ or $\dot{y}$ out of the Equation $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$; and the Fluent will give z the length of the Curve.

Z Ex. 1.

Ex. 1.

FIG. 73. *Let the Nature of the Curve be* $\frac{2}{3aa} \times \overline{aa+xx}|^{\frac{3}{2}} = y$.

Then it's Fluxion is $\dot{y} = \frac{2x\dot{x}}{aa}\sqrt{aa+xx}$: Then $\dot{z} = \sqrt{\dot{x}^2+\dot{y}^2} = \dot{x}\sqrt{1+\frac{4xx}{aa}+\frac{4x^4}{a^4}} = \dot{x}+\frac{2x^2\dot{x}}{aa}$: And (by Form the 1st) the Fluent $z = x+\frac{2x^3}{3aa}$, which needs no Correction.

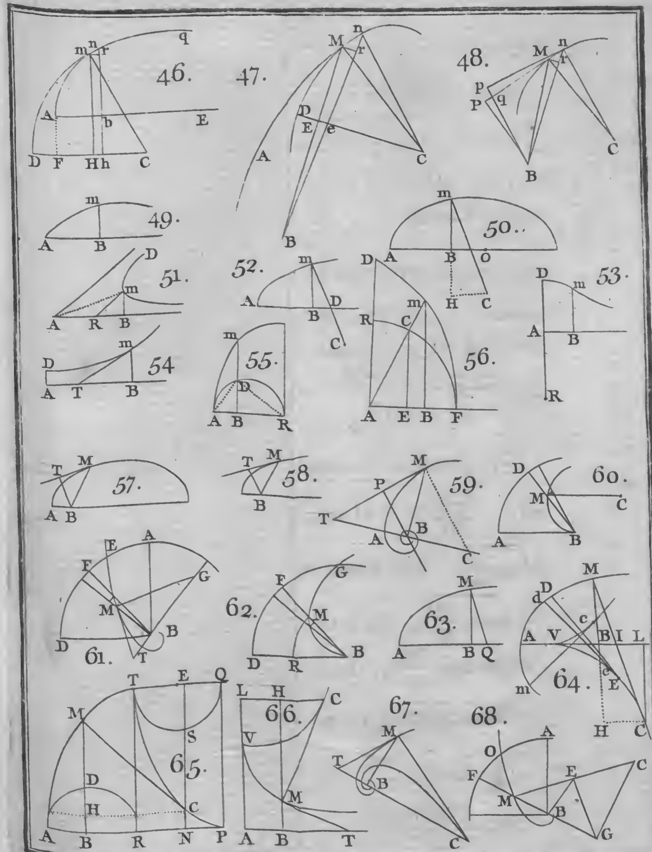
Ex. 2.

74. *In the common Parabola* $ax=yy$. Here $\dot{x} = \frac{2y\dot{y}}{a}$, whence $\dot{z} = \sqrt{\dot{x}^2+\dot{y}^2} = \frac{\dot{y}}{a}\sqrt{aa+4yy}$: And by Form the 9th and 13th, the corrected Fluent $z = \frac{y}{a}\sqrt{\frac{1}{4}aa+yy} + \frac{2.30258a}{4} \times \text{Log.} \frac{y+\sqrt{\frac{1}{4}aa+yy}}{\frac{1}{2}a}$.

Ex. 3.

74. *In the semicubical Parabola* $ax^2=y^3$, or $x = \frac{y^{\frac{3}{2}}}{a^{\frac{1}{2}}}$, and $\dot{x} = \frac{3y^{\frac{1}{2}}\dot{y}}{2a^{\frac{1}{2}}}$: And therefore $\dot{z} = \sqrt{\dot{x}^2+\dot{y}^2} = \dot{y}\sqrt{1+\frac{9y}{4a}}$, whose Fluent, by Form the 3d, is $z = \frac{8a}{27} \times \overline{1+\frac{9y}{4a}}|^{\frac{3}{2}}$. But in the Vertex, y and $z=0$; (whence $0 = \frac{8a}{27}$); therefore (Prob. XII. Sect. I.) the Fluent corrected is $z = \frac{8a}{27} \times \overline{1+\frac{9y}{4a}}|^{\frac{3}{2}} - \frac{8a}{27}$.

Ex. 4.



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Ex. 4.

Let $ax^m = y^{m+1}$ *be an Equation for various Parabolas.* FIG.

Then $x = \frac{y^{\frac{m+1}{m}}}{a^{\frac{1}{m}}}$, and $\dot{x} = \frac{m+1}{ma^{\frac{1}{m}}}y^{\frac{1}{m}}\dot{y}$, and $\dot{x}^2 =$ 74.

$\overline{\frac{m+1}{ma^{\frac{1}{m}}}}\Big|^2 y^{\frac{2}{m}}\dot{y}^2 = by^{\frac{2}{m}}\dot{y}^2$ (putting $b = \overline{\frac{m+1}{ma^{\frac{1}{m}}}}\Big|^2$). Whence $\dot{z} = \dot{y}\sqrt{1+by^{\frac{2}{m}}} = y^{\frac{1}{m}}\dot{y}\sqrt{b+y^{-\frac{2}{m}}}$. Now the Fluent of $y^{\frac{1}{m}}\dot{y}\sqrt{b+y^{-\frac{2}{m}}}$ will be had by Form the 15th, in finite Terms, when m is any positive even Number.

And likewise the Fluent of $\dot{y}\sqrt{1+by^{\frac{2}{m}}}$ will be found by Form the 11th, when m is any odd Number; finding first the Fluent of $y^{\frac{1}{m}-1}\dot{y}\sqrt{1+by^{\frac{2}{m}}}$ by Form the 9th and 13th.

Ex. 5.

To find the Length of the Arch of the Circle AM. 75.
Let Radius $AC = r$. Sine $BM = y$. $AB = x$, by the Nature of the Circle $2rx - xx = yy$, and $\dot{x} = \frac{y\dot{y}}{r-x} = \frac{y\dot{y}}{\sqrt{rr-yy}}$, therefore $\dot{z} = \sqrt{\dot{x}^2+\dot{y}^2} = \frac{r\dot{y}}{\sqrt{rr-yy}}$; and, by Form the 16th, $z = y + \frac{y^2}{2.3rr}A + \frac{3.3yy}{4.5rr}B + \frac{5.5.yy}{6.7.rr}C$ &c. where A, B, C, &c are the whole foregoing Terms.

Otherwise thus:

Let $AC = r$, Tangent $AT = t$, $Tt = \acute{t}$, $AM = z$, 76.
$Mn = \dot{z}$, then $CT = \sqrt{rr+tt}$. By similar Triangles CAT and Ttr, $Tr = \frac{r\acute{t}}{\sqrt{rr+tt}}$; and by the similar

 Triangles,

Triangles, CMn, CTr, $\dot{z} = \frac{rr\dot{t}}{rr+tt}$, or $\dot{z} = \frac{rr\dot{t}}{rr+tt}$ $= \dot{t} - \frac{t^2\dot{t}}{rr} + \frac{t^4\dot{t}}{r^4} - \frac{t^6\dot{t}}{r^6}$ &c. Whence (by Form the 16th) the Fluent $z = t - \frac{t^3}{3rr} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8}$ &c.

Now if $r=1$, and Arch $AM=30°$, then $t=\sqrt{\frac{1}{3}}$: And then $z=\sqrt{\frac{1}{3}} \times : 1 - \frac{1}{3 \cdot 3} + \frac{1}{5 \cdot 3^2} - \frac{1}{7 \cdot 3^3} + \frac{1}{9 \cdot 3^4} -$ &c. $= \frac{1}{12}$ the Circumference: And half the Circumference $= \frac{\sqrt{12}}{1} - \frac{\frac{1}{3}A}{3} - \frac{\frac{1}{3}B}{5} - \frac{\frac{1}{3}C}{7} - \frac{\frac{1}{3}D}{9}$ &c; where A, B, C, &c. are the preceding Numerators with their Signs.

Or thus, by collecting every two Terms into one, we shall have half the Circumference $= \frac{16}{3}\sqrt{3} \times$ into $\frac{1}{1 \cdot 3} + \frac{2}{5 \cdot 7 \cdot 9} + \frac{3}{9 \cdot 11 \cdot 9^2} + \frac{4}{13 \cdot 15 \cdot 9^3}$ &c; that is putting $\frac{16}{3}\sqrt{3}=A$, $\frac{1}{9}A=B$, $\frac{1}{9}B=C$, $\frac{1}{9}C=D$ &c; then half the Circumference $= \frac{1}{1 \cdot 3}A + \frac{2}{5 \cdot 7}B + \frac{3}{9 \cdot 11}C + \frac{4}{13 \cdot 15}D + \frac{5}{17 \cdot 19}E + \frac{6}{21 \cdot 23}F +$ &c. Which series will be as follows.

FIG.

$$\begin{array}{r}
3,0792014356780040773 8\\
58651455917676268 14\\
34558938672031471 1\\
2599304789007495 3\\
217949962958574 8\\
19433495662006 8\\
1802596675809 2\\
171866837038 7\\
16721676027 9\\
1652381938 6\\
165300677 6\\
16701774 9\\
1701478 0\\
174540 5\\
18010 6\\
1868 0\\
194 6\\
20 4\\
2 2\\
2\\
\hline
3,14159265358979323846 \text{ \&c}
\end{array}$$

= half the Circumference.

Ex. 6.

Let FMD be the Quadrant of an Ellipsis, $AD=a$, $AF=c$, $AB=x$, $BM=y$, $FM=z$. Then by the Nature of the Curve $y=\frac{c}{a}\sqrt{aa-xx}$, and $\dot{y}=\frac{-cx\dot{x}}{a\sqrt{aa-xx}}$; whence $\dot{z}=\sqrt{\dot{x}^2+\dot{y}^2}=\frac{\dot{x}\sqrt{a^4-aaxx+ccxx}}{a\sqrt{aa-xx}}$ $=$(putting $d=\frac{aa-cc}{aa}$) $\frac{\dot{x}\sqrt{aa-dxx}}{\sqrt{aa-xx}}=\frac{\dot{x}\sqrt{a^2-dxx\times aa-xx}}{aa-xx}$ $=\dot{x}+\frac{1-d}{2aa}x^2\dot{x}+\frac{3-2d-dd}{8a^4}x^4\dot{x}$ &c; whence $z=x+\frac{1-d}{6aa}x^3+\frac{3-2d-dd}{40a^4}x^5+\frac{5-3d-dd-d^3}{112a^6}x^7+$ &c, for the Arch FM. 77.

But

FIG. But for the Quadrant FD, $\dot{z}=\frac{\dot{x}\sqrt{aa-dxx}}{\sqrt{aa-xx}}=\frac{\dot{x}}{\sqrt{aa-xx}}\times : a-\frac{dxx}{2a}-\frac{ddx^4}{2.4a^3}-\frac{3d^3x^6}{2.4.6a^5}$ &c, but the Fluent of $\frac{\dot{x}}{\sqrt{aa-xx}}=$ Arch of the circumscribing Quadrant of the Circle divided by $a=\frac{Q}{a}$; whence the Quadrant FD or $z=Q\times 1-\frac{d}{2.2}+\frac{3d}{4.4}A+\frac{3.5d}{6.6}B+\frac{5.7d}{8.8}C+$ &c, by Form 17th.

Ex. 7.

78. *To find the Length of the Arch of the Hyperbola.* Let $a=$ half the Transverse, $c=AF$ half the conjugate, $AB=x$, $BM=y$: Then $y=\sqrt{cc+\frac{cc}{aa}xx}$, and $\dot{y}=\frac{cx\dot{x}}{\sqrt{a^4+aaxx}}$, and $\dot{x}^2+\dot{y}^2=\frac{aa+\frac{aa+cc}{aa}xx}{aa+xx}\dot{x}^2$; put $d=\frac{aa+cc}{aa}$, and then $\dot{z}=\dot{x}\sqrt{\frac{aa+dxx}{aa+xx}}=\dot{x}\sqrt{1-\frac{1-d}{aa}x^2-\frac{1-d}{a^4}x^4-\frac{1-d}{a^6}x^6}=\dot{x}\times : 1-\frac{1-d}{2aa}x^2+\frac{3-2d-dd}{8a^4}x^4-\frac{5-3d-dd-d^3}{16a^6}x^6+$ &c; therefore $z=x-\frac{1-d}{6aa}x^3+\frac{3-2d-dd}{40a^4}x^5-\frac{5-3d-dd-d^3}{112a^6}x^7$ &c for the Arch FM.

Ex. 8.

79. *Let AM be the Cycloid*, $AB=x$, $BM=y$, $AR=a$, $BD=v$. Arch $AD=s$, then $y=s+v$, by the Nature of the Circle $vv=ax-xx$, and $\dot{v}=\frac{a\dot{x}-2x\dot{x}}{2v}$, and

$\dot{s}=$

$\dot{s} = \frac{a\dot{x}}{2v}$; therefore $\dot{y} = \dot{s} + \dot{v} = \frac{a-x}{v}\dot{x}$, whence $\dot{z} =$ FIG.
$\sqrt{\dot{x}^2+\dot{y}^2} = \dot{x}\sqrt{1 + \frac{\overline{a-x}^2}{vv}} = \frac{\dot{x}}{v}\sqrt{aa-ax} = \frac{\dot{x}\sqrt{aa-ax}}{\sqrt{ax-xx}}$
$= \dot{x}\sqrt{\frac{a}{x}}$. Therefore $z = 2\sqrt{ax} = 2$ Cord AD.

Ex. 9.

Let GM be the Log. Curve, $AB=x$, $BM=y$, $AG = b$, Subtangent $BR=a$, then $a\dot{y}=y\dot{x}$, therefore $\dot{z} = \sqrt{\dot{x}^2+\dot{y}^2} = \frac{\dot{y}}{y}\sqrt{aa+yy} = \frac{y\dot{y}}{\sqrt{aa+yy}} + \frac{aay^{-2}\dot{y}}{\sqrt{1+aay^{-2}}}$. 69.
Whence (by Form the 3d and 9th) $z = \sqrt{aa+yy} - 2.30258a \times \text{Log.}\frac{a+\sqrt{aa+yy}}{y}$; and corrected (by Prop. XII.) z or $GM = \sqrt{aa+yy} - \sqrt{aa+bb} + 2.302585a \times \text{Log.}\ \frac{ay+y\sqrt{aa+bb}}{ab+b\sqrt{aa+yy}}$.

Ex. 10.

In the Cissoid DM, let $AD=a$, $AB=x$, $BM=y$, 80.
then $y = \frac{\overline{a-x}|^2}{\sqrt{ax-xx}} = \frac{\overline{a-x}|^{\frac{3}{2}}}{x^{\frac{1}{2}}}$, and $\dot{y} = \frac{-a\dot{x}-2x\dot{x}}{2xx}\sqrt{ax-xx}$, whence $\dot{z} = \sqrt{\dot{x}^2+\dot{y}^2} = \frac{a\dot{x}}{2x}\sqrt{\frac{a+3x}{x}} = \frac{ax^{-\frac{3}{2}}\dot{x}}{2}\sqrt{a+3x}$, and (by Form the 9th, 13th and 12th) $z = 2.3025a\sqrt{3} \times \text{Log.}\sqrt{3x}+\sqrt{a+3x}: - a\sqrt{\frac{a+3x}{x}}$. And the Fluent corrected, z or $MD = a\sqrt{\frac{a+3x}{x}} - 2a + 2.30258a\sqrt{3}\times\text{Log.}\frac{\sqrt{3a}+\sqrt{4a}}{\sqrt{3x}+\sqrt{a+3x}}$.

Ex. 11.

Let AM be the Quadratrix, $AB=x$, $BM=y$, Arch $AK=t$, Sine $KG=v$, $AC=a$. Then (by Ex. the 5th) 81.
$\dot{t} =$

FIG. $t = v + \frac{v^3}{6aa} + \frac{3v^5}{40a^4} + \frac{5v^7}{112a^6} + \&c = y$, by the Nature of the Quadratrix. Then by Reversion of Series $v = y - \frac{y^3}{6aa} + \frac{y^5}{120a^4} - \frac{y^7}{5040a^6}$ &c; then $\sqrt{aa-vv} = a - \frac{yy}{2a} + \frac{y^4}{24a^3} - \frac{y^6}{720a^5}$ &c $=$ CG; and by similar Triangles $(KG)\ v : GG :: (MB)$ $y : (BC)\ a-x = a - \frac{y^2}{3a} - \frac{y^4}{45a^3} - \frac{2y^6}{945a^5}$ &c; whence $x = \frac{yy}{3a} + \frac{y^4}{45a^3} + \frac{2y^6}{945a^5}$ &c; and $\dot{x} = \frac{2y\dot{y}}{3a} + \frac{4y^3\dot{y}}{45a^3} + \frac{4y^5\dot{y}}{315a^5}$ &c; therefore $\dot{z} = \sqrt{\dot{x}^2+\dot{y}^2} = \dot{y} \times 1 + \frac{2y^2}{9aa} + \frac{14y^4}{405a^4} + \frac{604y^6}{127575a^6}$ &c, and $z = y + \frac{2y^3}{27a^2} + \frac{14y^5}{2025a^4} + \frac{604y^7}{893025a^6}$ &c.

Ex. 12.

82. *Let BM be Archimedes's Spiral, whose Equation is* $rv = cy$, putting $v =$ Arch ED. Then $r\dot{v} = c\dot{y}$, but (by similar Triangles) $\dot{v} = \frac{r\dot{x}}{y}$, whence $\dot{x} = \frac{cy\dot{y}}{rr}$: Therefore $\dot{z} = \sqrt{\dot{x}^2+\dot{y}^2} = \frac{\dot{y}}{rr} \times \sqrt{r^4+ccyy}$; and (by Form the 9th and 13th) the Fluent (corrected) is $z = \frac{y}{2rr}\sqrt{r^4+ccyy} + \frac{2.302585rr}{2c} \times \text{Log.}\frac{cy+\sqrt{r^4+ccyy}}{rr}$.

Ex. 13.

~~82.~~ 79. *Let BM be the Log. Spiral*, and let c, t, s, express the Ratio of the Tangent TM, Subtangent TB, and Ordinate BM. Then $\dot{z} = \frac{c\dot{y}}{s}$, and the Fluent $z = \frac{cy}{s} = TM$. Therefore the Length of the Spiral MB making an infinite Number of Revolutions is equal to the Tangent MT.

Ex. 14

Ex. 14.

Let BM be the hyperbolic spiral, $AB=a$, Arch $Ao=b$, Arch $AD=v$, $MB=y$, then $ab=vy$, and thence $v\dot{y}+y\dot{v}=0$, and $y\dot{v}=-v\dot{y}=\frac{-ab\dot{y}}{y}$; and by similar Triangles $-y\dot{v}=a\dot{x}$, and thence $\dot{x}=\frac{b\dot{y}}{y}$; therefore $\dot{z}=\sqrt{\dot{x}^2+\dot{y}^2}=\frac{\dot{y}}{y}\sqrt{bb+yy}=\frac{bby^{-2}\dot{y}}{\sqrt{1+bby^{-2}}}+\frac{y\dot{y}}{\sqrt{bb+yy}}$. And the Fluent (by Form 3 and 9) is $z=\sqrt{bb+yy}-b\times 2.\ 302585$ Log. $\frac{b+\sqrt{bb+yy}}{y}$: Whence taking any Ordinate $BQ=d$, the Length of the Arch $QM=\sqrt{bb+dd}-\sqrt{bb+yy}+2.\ 30258b\times$ Log. $\frac{by+y\sqrt{bb+dd}}{bd+d\sqrt{bb+yy}}$. FIG. 83.

PROB. IX.

To transform Spirals into geometrical Curves, or geometrical Curves into Spirals, of equal Length.

Let *ADd* be a geometrical Curve related to the Axis *AB*; let the perpendicular Ordinates *BD*, *bd*, be infinitely near together; and let the Points *B b* be supposed to approach one another, and to coincide in *B*, whilst the Particle of the Curve *Dd* remains the same; then the Parallelogram *BDdb* will be changed into the triangle *BDd*. In like manner let all the Points in *AB* be supposed to be contracted into one Point *B*, and then the Figure *ADd* will be transformed into the Spiral *QDd*, whose Center is *B*, in which the corresponding Ordinates are equal, and the Triangles 84.

FIG, Ddr in both Figures similar and equal; and the Area of the Figure ABD twice the Area of the spiral BQD.

85. Again let ADd be a Spiral, ABD it's Complement; BD, bd, two Arches of Circles infinitely near each other, whose Center is A; AH a geometrical Curve (to the Abscissa $AB=AD$) equal in Length to the Spiral: Let BH, bh, be drawn infinitely near each other, and HK parallel to AB. Then the Triangles DdC and HbK are similar and equal, therefore $dC=Kb$. Now let AB or $AD=y$, $BH=u$, Arch $BD=v$, and by similar Triangles $y:v::Ab:bC::Bb:bC-v$ $=\frac{v}{y}\times Bb$. Therefore $bC+Cd-v$ or $bd-v=Kb+\frac{v}{y}\times Bb$: therefore (because the Moments are as the Fuxions) $\dot{v}=\dot{u}+\frac{v\dot{y}}{y}$.

84. 1. *Therefore in Curves referred to an Axis, let* $AB=x$, $BD=y$, *Radius* BR *or* $BT=b$, *Arch* $RT=v$, $Dr=\dot{x}$, $rd=\dot{y}$, $TS=\dot{v}$. *In the Curve* AD, *expunge* x *and* $\dot{x}$ *out of the Equation of the Curve, by help of the Equation* $b\dot{x}=y\dot{v}$; *and the Fluent will give the Nature of the Spiral. Likewise in the Spiral* QD, *expunge all Quantities except* x *and* y *and their Fluxions out of the Equation* $b\dot{x}=y\dot{v}$, *by Help of the Equation of the Curve; and the Fluent will shew the Nature of the geometrical Curve* AD.

85. 2. *In the Spiral* AD, *where there is given the Relation of* AD (y) *to the Arch* BD (v): *By Help of the Equation of the Spiral, exterminate* v *and* $\dot{v}$ *out of the Equation* $\dot{v}=\dot{u}+\frac{v\dot{y}}{y}$, *and the Fluent gives the Nature of the Curve* AH. *Or, by the Equation of the Curve* AH, *expunge* u *and* $\dot{u}$ *out of the Equation* $\dot{v}=\dot{u}+\frac{v\dot{y}}{y}$, *and the Fluent gives the Nature of the Spiral.*

Ex. 1.

Example 1.

Let $ax = yy$ *denote a Parabola ADd,* then we have FIG. 84.
$\dot{x} = \frac{2y\dot{y}}{a} = \frac{y\dot{v}}{b}$, and $2b\dot{y} = a\dot{v}$; whence the Fluent $2by = av$; therefore the Spiral *QD* is that of *Archimedes.*

Ex. 2.

Let AD be a Circle, $2ax - xx = yy$, Arch $AD = z$, 86.
therefore $\dot{x} = \frac{y\dot{y}}{a-x} = \frac{y\dot{y}}{\sqrt{aa-yy}} = \frac{y\dot{v}}{b}$, therefore $\dot{v} = \frac{b\dot{y}}{\sqrt{aa-yy}}$, let $b = a$, then $\dot{v} = \frac{a\dot{y}}{\sqrt{aa-yy}} = \dot{z}$, therefore $v = z$, that is $RT = AD$, and $BD = TF$; draw GD, then since $BG = BT$, and $BD = TF$, and $\angle DBG = \angle FTB$, therefore $\angle GDB = \angle BFT$ = a Right-angle, and the Curve BDG is a Semi-circle.

Ex. 3.

Let $aa = xy$ *be the Equation of an equilateral Hyperbola between the Assymptotes,* then $x\dot{y} + y\dot{x} = 0$, and 87.
$\dot{x} = \frac{-x\dot{y}}{y} = -\frac{aa}{yy}\dot{y} = \frac{y\dot{v}}{b}$. Therefore $-\frac{baa\dot{y}}{y^3} = \dot{v}$, and thence $v = \frac{baa}{2yy}$, or putting $b = 2a$, $a^3 = vyy$, for the Nature of the Spiral *QDd.*

Ex. 4.

Let ED be the logarithmic Curve, whose Equation 88.
is $y\dot{x} = a\dot{y}$, then $\dot{x} = \frac{a\dot{y}}{y} = \frac{y\dot{v}}{b}$, and $\frac{ab\dot{y}}{yy} = \dot{v}$, therefore $v = -\frac{ab}{y}$: Therefore any Portion of the Arch RT or v is $= ab \times \frac{DB - EA}{DB \times EA}$; BD, EA being the corresponding Ordinates.

Ex. 5.

FIG. *Let $ay=bx$ denote a plain Triangle*, then $a\dot{y}=b\dot{x}=y\dot{v}$, therefore $\frac{a\dot{y}}{y}=\dot{v}$, consequently $v=a\times\text{Log.}\,y$ an Equation to the Log. Spiral.

Ex. 6.

84. *Let QD be the Spiral of Archimedes*, whose Equation is $dy=av$; then $\frac{d\dot{y}}{a}=\dot{v}=\frac{b\dot{x}}{y}$, or $dy\dot{y}=ab\dot{x}$, and the Fluent $\frac{dyy}{2}=abx$, or $yy=\frac{2ab}{d}x$, an Equation to a Parabola.

85. Or thus: *Let $AB=y$, $BD=v$*, then $av=yy$, whence $\frac{2y\dot{y}}{a}=\dot{v}=\dot{u}+\frac{v\dot{y}}{y}=\dot{u}+\frac{y\dot{y}}{a}$, and $\dot{u}=\frac{y\dot{y}}{a}$; therefore $2au=yy$, an Equation to the common Parabola, as before.

Ex. 7.

84. *Let QD be the logarithmic Spiral*, whose Equation is $a\dot{y}=b\dot{x}$, therefore $ay=bx$ a plain Triangle.

Ex. 8.

88. *Let QD be the hyperbolic Spiral*, then $ab=vy$, and $v\dot{y}+y\dot{v}=0$, or $\dot{v}=\frac{-v\dot{y}}{y}=\frac{-ab\dot{y}}{yy}=\frac{b\dot{x}}{y}$, or $\frac{-a\dot{y}}{y}=\dot{x}$, whence the Fluent $-\frac{x}{a}=\text{Log.}\,y$ an Equation to the Log. Curve.

Ex. 9.

85. *Suppose $y^3=av^2$ to denote the Spiral ADd*, then $\dot{v}=\frac{3y^{\frac{1}{2}}\dot{y}}{2\sqrt{a}}=\dot{u}+\frac{v\dot{y}}{y}=\dot{u}+\frac{y^{\frac{1}{2}}\dot{y}}{\sqrt{a}}$; therefore $\dot{u}=\frac{y^{\frac{1}{2}}\dot{y}}{2\sqrt{a}}$, whence $u=\frac{y^{\frac{3}{2}}}{3\sqrt{a}}$, or $9av^2=y^3$, a semicubical Parabola.

Ex. 10.

Ex. 10.

If $v = y\sqrt{\frac{a+y}{c}}$ *be the Equation of the Spiral*; then $\dot{v} = \frac{2a\dot{y}+3y\dot{y}}{2\sqrt{ca+cy}} = \dot{u} + \frac{v\dot{y}}{y} = \dot{u} + \dot{y}\sqrt{\frac{a+y}{c}} = \dot{u} + \frac{a\dot{y}+y\dot{y}}{\sqrt{ca+cy}}$; whence $\dot{u} = \frac{y\dot{y}}{2\sqrt{ca+cy}}$, therefore $u = \frac{y-2a}{3c}\sqrt{ca+cy}$. FIG. 85.

Ex. 11.

Let BDdE be a Semicircle, BD=y, BE=a, by similar Triangles $Dr = \frac{BD \times rd}{ED}$, or $\dot{x} = \frac{y\dot{y}}{\sqrt{aa-yy}}$, whence $x = -\sqrt{aa-yy}$; when corrected, $x = a - \sqrt{aa-yy} = BE - ED$, therefore ADL is the Quadrant of a Circle, whose Radius is a. 89.

Ex. 12.

Let QDd be a Parabola, B the Focus a = Latus rectum, $BD=y$, DP a Tangent: By the Nature of the Figure the Perpendicular $BP = \frac{1}{2}\sqrt{ay}$, and by similar Triangles $Dr = \frac{BP \times dr}{PD}$, or $\dot{x} = \frac{a\dot{y}}{\sqrt{4ay-aa}}$; therefore $x = \sqrt{ay-\frac{1}{4}aa}$ = Ordinate DI: Hence the Curve will be the same Parabola referred to the Axis AB; making $AQ = QB$, and AB perpendicular to AQ. 90.

PROB.

PROB. X.

To find the Areas of Curves.

FIG. 91. In any Curve AD related to the Abſciſſa AB, produce the Ordinate DB till BE be equal to a given Line, and compleating the Parallelogram $ABEC$, if the Areas $ACEB$ and ADB be conceived to be generated by the right Lines BE, BD, moving along the Abſciſſa AB; then the Fluxions of the Areas $ACEB$ and ABD will be as the deſcribing Lines BE, BD, drawn into their Velocities of moving, that is into the Fluxions of the Abſciſſas: Now ſince $BE \times AB =$ Area $ACEB$, and therefore the Fluxion of the Area $ACEB$ $= BE \times$ Fluxion of AB; conſequently the Fluxion of the Area $ABD = DB \times$ Fluxion of AB.

92. In like Manner in Curves related to a fixed Point B, the fluxionary Triangle BDM is the Moment of the Area, and this is $= BM \times \frac{DR}{2}$, or $=$ the Perpendicular BT (on the Tangent at M) $\times$ by $\frac{1}{2}DM$; but DR and DM are as their generating Fluxions. Hence,

91. 1. *In Curves related to an Axis AB, let $AB = x$, $BD = y$, Area $ABD = z$, then $\dot{z} = y\dot{x}$; therefore by the Equation of the Curve, expunge one of the Quantities y or $\dot{x}$ out of the Equation $\dot{z} = y\dot{x}$, and finding the Fluent it gives the Value of z the Area. Sometimes it will be neceſſary to find the Area of the Complement AbD, but the Rule will ſtill be the ſame if you make Ab the Abſciſſa.*

Note, if the Ordinates are not at Right-angles to the Abſciſſa, the Area before found muſt be diminiſhed in the Ratio of Radius to the Sine of the true Angle.

2. *In*

2. *In Curves related to a fixed Point B, let $BD=y$, Perpendicular BT (on the Tangent) $=p$. Curve $AD =v$, $DR=\dot{x}$, Area $BAD=z$: Then by the Nature of the Curve expunge y or $\dot{x}$ out of the Equation $\dot{z}=\frac{y\dot{x}}{2}$; or expunge p or $\dot{v}$ out of the Equation $\dot{z}=\frac{p\dot{v}}{2}$; and the Fluent will give z the Area.* FIG. 92.

And if the fluxionary Triangle BDM can be computed any other way, and the heterogeneous Quantities expunged by the Equation of the Curve; the Fluent will give the Area as before.

Ex. 1.

To find the Area of a Triangle. Let the Base $CD=b$, Perpendicular $AQ=p$, $AB=x$, dD (parallel to CD) $=y$: By similar Triangles $y=\frac{bx}{p}$, therefore $\dot{z}=y\dot{x} =\frac{bx\dot{x}}{p}$, therefore the Fluent $z=\frac{bxx}{2p}=\frac{xy}{2}$; and when $x=p$, and $y=b$, the Area $z=\frac{bp}{2}$. Or since $\dot{x}=-\frac{p\dot{y}}{b}$, therefore $\dot{z}=y\dot{x}=\frac{py\dot{y}}{b}$, and $z= \frac{pyy}{2b}=\frac{xy}{2}$, as before. 93.

Otherwise thus;

Let the Perpendicular $BC=p$, $BD=y$, $de=\dot{y}$; and by similar Triangles $Dd=\frac{y\dot{y}}{\sqrt{yy-pp}}$, therefore $\dot{z}= \frac{py\dot{y}}{2\sqrt{yy-pp}}=$ Area BDd, and $\dot{z}=\frac{py\dot{y}}{\sqrt{yy-pp}}$, and thence $z=\frac{1}{2}p\sqrt{yy-pp}$. Now in the Point C, y is a Minimum $=p$. Therefore by the Schol. Prop. XII. first the Part ABC must be found, which is $\frac{p}{2}\sqrt{AB^2-pp}$, and then the Part BCD, which will be $\frac{p}{2}\times\sqrt{BD^2-pp}$. 94.

And

FIG.

And the whole $ABD=\frac{p}{2}\sqrt{AB^2-pp}+\frac{p}{2}\sqrt{BD^2-pp}$ or $\frac{1}{2}AD\times CB$.

Ex. 2.

91. *Let ABD be any Parabola, where* $x=y^m$; then $\dot{x}=my^{m-1}\dot{y}$, therefore $\dot{z}=y\dot{x}=my^m\dot{y}$; and $z=\frac{m}{m+1}y^{m+1}=\frac{m}{m+1}yx$, and if $m=2$, $z=\frac{2}{3}xy$.

Or thus: *Let* $Ab=x$, $bD=y$, $x=y^{\frac{1}{m}}$, then $\dot{x}=\frac{1}{m}y^{\frac{1}{m}-1}\dot{y}$, therefore $\dot{z}=y\dot{x}=\frac{1}{m}y^{\frac{1}{m}}\dot{y}$, and thence $z=\frac{1}{m+1}y^{\frac{1}{m}+1}=\frac{1}{m+1}xy$: And if $m=2$, $z=\frac{1}{3}xy$, the Area AbD.

Ex. 3.

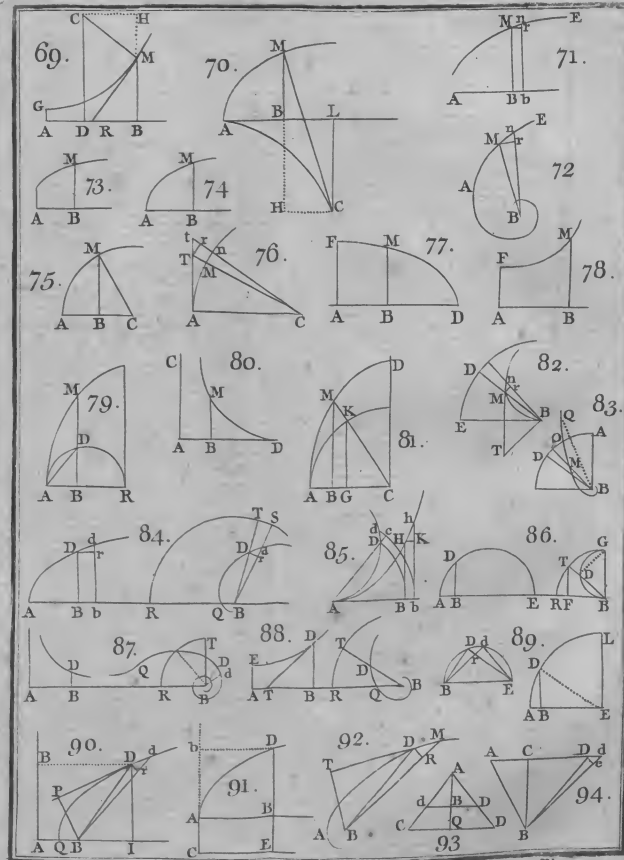
95. *Let FD be an Hyperbola,* $CA=a$, $AF=b$, $AB=x$, $BD=\frac{ab}{a+x}=y$. Then $\dot{z}=y\dot{x}=\frac{ab\dot{x}}{a+x}=b\dot{x}-\frac{bx\dot{x}}{a}+\frac{bx^2\dot{x}}{aa}-\frac{bx^3\dot{x}}{a^3}$ &c, and the Area $ABDF$ or $z=bx-\frac{bx^2}{2a}+\frac{bx^3}{3aa}-\frac{bx^4}{4a^3}+\frac{bx^5}{5a^4}-$ &c. Or (by Form the 4th) the Fluent (corrected) is $z=2.302585ab\times\text{Log.}\frac{a+x}{a}$.

Ex. 4.

95. *Let FD be any kind of Hyperbola,* $CB=x$, $BD=y$, and suppose $y=\frac{1}{x^n}$, then $\dot{z}=y\dot{x}=x^{-n}\dot{x}$; and the Area $z=\frac{1}{1-n}x^{1-n}=\frac{1}{1-n}xy$, for the Area CBD. If the Angle ABD is oblique, that Area must be diminished in the Ratio of Radius to the Sine of ABD.

If $n=1$ the Space will be infinite, if n be greater than 1, you get the Space BDE.

Ex. 5.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89.
90.
91.
92.
93
94.

p. 184. IV. p.

Ex. 5.

Let AD be a Circle, $AC=a$, $AB=x$, $BD=y=$ FIG. 96.
$\sqrt{ax-xx}$, then $\dot{z}=y\dot{x}=\dot{x}\sqrt{ax-xx}$, whose Fluent (by Form 16th) is $z=\sqrt{ax}\times:\frac{2}{3}x-\frac{x^2}{5a}-\frac{x^3}{4.7a^2}$
$-\frac{1.3x^4}{4.6.9a^3}-\frac{3.5.x^5}{4.6.8.11a^4}$ &c $=\frac{2x\sqrt{ax}}{3}-$
$\frac{\frac{x}{2a}A}{5}+\frac{\frac{x}{4a}B}{7}+\frac{\frac{3x}{6a}C}{9}+\frac{\frac{5x}{8a}D}{11}$ &c; where A, B, C, &c, are the preceding Numerators. Or (by Form 10th, 11th, 13th) $z=\frac{aa}{8}\phi+\frac{2x-a}{4}$
$\sqrt{ax-xx}$ (putting $\phi=,01745\times$ twice the Degrees in the Arch whose Sine is $\sqrt{\frac{x}{a}}$, and Radius 1.)

If AD be 60°, then $x=\frac{1}{4}a$, and Area $ABD=\frac{1}{12}aa$
$-\frac{3}{2.4.5}A+\frac{1.5}{4.4.7}B+\frac{3.7}{6.4.9}C+\frac{5.9}{8.4.11}D$,
to which adding the Triangle $DBE=\frac{aa}{32}\sqrt{3}$, you have $\frac{1}{6}$ the Area of the Circle. Here A, B, C, &c, are the preceding Terms.

If $y=\sqrt{ax+xx}$ be an Equation to a right-angled Hyperbola, we shall in like manner find $z=\sqrt{ax}\times:$
$\frac{2}{3}x+\frac{x^2}{5a}-\frac{x^3}{4.7aa}+\frac{3.x^4}{4.6.9a^3}-\frac{3.5x^5}{4.6.8.11a^4}$
$+$ &c, or $z=\frac{a+2x}{4}\sqrt{ax+xx}-\frac{aa}{8}\phi$, where $\phi=$
2.30258 Log. $\frac{a+2x+2\sqrt{ax+xx}}{a}$.

Likewise the Area of an Ellipsis is found after the same manner as the Circle. Or if you put c for the Conjugate, it is no more than multiplying the former Area of the Circle by $\frac{c}{a}$. And thus $\frac{c}{a}$ multiplied

FIG. into the former Area of the Hyperbola, gives the Area of the Hyperbola whose Conjugate is c.

Ex. 6.

97. *In the Ellipsis FDE*, to find the Area adjoining to the Center A. Let $AE=a$, $AF=c$, $AB=x$, $BD=y$, then $y=\frac{c}{a}\sqrt{aa-xx}$, by the Nature of the Figure; then $\dot{z}=y\dot{x}=\frac{c\dot{x}}{a}\sqrt{aa-xx}$, whence (by Form 16th) $z=\frac{cx}{1}-\frac{\frac{xx}{2aa}A}{3}+\frac{\frac{xx}{4aa}C}{5}+\frac{\frac{3xx}{6aa}D}{7}+\frac{\frac{5xx}{8aa}E}{9}$ &c for the Area $ABDF$. Or $z=\frac{ca\varphi}{2}+\frac{cx}{2a}\sqrt{aa-xx}$, putting $\varphi=,017453\times$ Degrees in the Arch whose Sine is $\frac{x}{a}$, and Radius 1.

98. After the same Manner in the Hyperbola, when $y=\frac{c}{a}\sqrt{aa+xx}$, we find $z=\frac{cx}{1}+\frac{\frac{xx}{2aa}A}{3}-\frac{\frac{xx}{4aa}B}{5}-\frac{\frac{3xx}{6aa}C}{7}-\frac{\frac{5xx}{8aa}D}{9}$ &c. Or $z=\frac{ca\varphi}{2}+\frac{cx}{2a}\sqrt{aa+xx}$, putting $\varphi=2.30258\,\text{Log.}\frac{x+\sqrt{aa+xx}}{a}$.

Ex. 7.

99. *To find the Area ADB of the Ellipsis ADE, generated by the Line BD revolving round the focus B.*

Let $2a=$ Transverse AE, $2c=$ Conjugate, $BD=y$, draw the Tangent DT and BT perpendicular to it; then by the conic Sections $BT=\frac{cy}{\sqrt{2ay-yy}}$, whence $DT=\frac{y\sqrt{2ay-yy-cc}}{\sqrt{2ay-yy}}$. And by similar Triangles $DT:TB::MR:RD$, that is $\sqrt{2ay-yy-cc}:c::\dot{y}$

$\dot{y} : \dot{x} = \frac{c\dot{y}}{\sqrt{2ay-yy-cc}}$; whence $\dot{z} = \frac{y\dot{x}}{2} =$ FIG.

$\frac{cy\dot{y}}{2\sqrt{-cc+2ay-yy}}$: Put $v = a - y$, $p = aa - cc$;

and (by Form 27th) $\dot{z} = \frac{cv\dot{v}}{2\sqrt{p-vv}} - \frac{ca\dot{v}}{2\sqrt{p-vv}}$;

whence (by Forms 3d and 8th) $z = \frac{-c}{2}\sqrt{y-vv} -$

$\frac{ca}{2} \times ,01745$ Degrees in the Arch whose Sine is $\sqrt{\frac{vv}{p}}$, but when $z = 0$, $y = a - \sqrt{p}$, therefore correcting the Fluent, and putting $\varphi =$ Number of Degrees in the Arch whose Co-sine is $\frac{a-y}{\sqrt{aa-cc}}$, and Radius 1, then we have the Area *BAD* or $z = \frac{,0174533ca\varphi}{2} - \frac{c}{2}\sqrt{-cc+2ay-yy}$. And the Area of the Semi-ellipsis $AME = ,01745ca \times 90 = \frac{1}{2}ca \times 3{\cdot}141592$.

Ex. 8.

To find the parabolic Area BAD, generated by the line BD revolving round the Focus B. 100.

Let $a =$ Latus Rectum, $BD = y$; by the Nature of the Parabola the perpendicular $BT = \frac{1}{2}\sqrt{ay} = p$, and by similar Triangles $DT : DB :: MR : MD$, or $\sqrt{yy - \frac{1}{4}ay} : y :: \dot{y} : \dot{v} = \frac{y\dot{y}}{\sqrt{yy-\frac{1}{4}ay}}$; whence $\dot{z} = \frac{p\dot{v}}{2} = \frac{\sqrt{a}}{4} \times \frac{y\dot{y}}{\sqrt{y-\frac{1}{4}a}}$; and (by Form 4th and 11th) the Fluent $z = \frac{a+2y}{12}\sqrt{ay - \frac{1}{4}aa}$.

Ex. 9.

Let AD be an Hyperbola, B the Center, $BA = a$, Semi-conjugate $= b$, $BC = x$, Tangent $AT = t$, Ordinate $CD = y$; by similar Triangles $ay = tx$, and by the Nature of the Figure $bbxx - bbaa = aayy = ttxx$, and 101.

FIG. $xx = \frac{bbaa}{bb-tt}$. The fluxionary triangle BTt is to the fluxionary triangle BDd, as BT^2 to BD^2 or aa to xx, that is $\frac{a\dot{t}}{2} : \dot{z} :: aa : \frac{bbaa}{bb-tt} :: bb - tt : bb$; whence $\dot{z} = \frac{\frac{1}{2}bba\dot{t}}{bb-tt}$, and the Fluent $z = \frac{at}{2} + \frac{at^3}{6bb} + \frac{at^5}{10b^4} + \frac{at^7}{14b^6}$ *&c*: Or (by Form the 6th) the Fluent $z = \frac{1}{4}ab \times 2.30258 \text{ Log.} \frac{b+t}{b-t}$, for the hyberbolic Sector BAD.

Or thus: Since $t = (\frac{ay}{x} = \frac{ay}{\sqrt{aa + \frac{aa}{bb}yy}} =) \frac{by}{\sqrt{bb+yy}}$, therefore $\dot{t} = \frac{b^3\dot{y}}{\overline{bb+yy}|^{\frac{3}{2}}}$, therefore $\dot{z} = (\frac{\frac{1}{2}abb\dot{t}}{bb-tt} =) \frac{ab\dot{y}}{2\sqrt{bb+yy}}$; and the Fluent $z = \sqrt{bb+yy} \times : \frac{ay}{2b} - \frac{2yy}{3bb}A - \frac{4yy}{5bb}B - \frac{6yy}{7bb}C$ *&c*, by Form the 15th. Or (by Form the 9th) $z = \frac{1}{2}ab \times 2.30258 \times \text{Log.} \frac{y+\sqrt{bb+yy}}{b} =$ Sector BAD.

Ex. 10.

102. *Let RD be the Logarithmic Curve*, $AB = x$, $BD = y$, $AR = b$, Subtangent $= a$, by similar Triangles $a\dot{y} = y\dot{x} = \dot{z}$, whence $z = ay$, and (corrected) Area $ABDR = ay - ab$.

Ex. 11.

103. *Let AD be the Cissoid of Diocles*, $AE = a$, $AB = x$, $BD = y$, by the Nature of the Curve $y = \frac{x^{\frac{3}{2}}}{\sqrt{a-x}}$; therefore $\dot{z} = y\dot{x} = x^{\frac{3}{2}}\dot{x} \times \overline{a-x}|^{-\frac{1}{2}}$, and the Fluent (by Form the 10th and 11th) is $z = 3$ Sectors $\overline{CAM - \frac{3}{4}a - \frac{1}{2}x\sqrt{ax-xx}}$, that is $z = 3\,AMB - 2x\sqrt{ax-xx}$;

Ex. 12.

Ex. 12.

Let ED be the Concoid of Nichomedes, CA=b, AE FIG.
$=a$, $AB=x$, $BD=y$; the Equation of the Curve is 104.
$\overline{b+y}\sqrt{aa-yy}=xy$, in Fluxions $\dot{y}\sqrt{aa-yy}-\overline{b+y}\times y\dot{y}\times\overline{aa-yy}^{-\frac{1}{2}}=x\dot{y}+y\dot{x}$, therefore $\dot{z}=y\dot{x}=\dot{y}\sqrt{aa-yy}-y\dot{y}\times\frac{b+y}{\sqrt{aa-yy}}\;(-x\dot{y}=)-\frac{b+y}{y}\dot{y}\sqrt{aa-yy}=\frac{-y^2\dot{y}}{\sqrt{aa-yy}}-\frac{baa\dot{y}}{y\sqrt{aa-yy}}=\frac{-y^2\dot{y}}{\sqrt{aa-yy}}-\frac{ba^2y^{-2}\dot{y}}{\sqrt{-1+a^2y^{-2}}}$; whose corrected Fluent (by Form 10 and 11, and Form 9th) is $z=\frac{1}{2}y\sqrt{aa-yy}+2.30258ab\times$ Log. $\frac{a+\sqrt{aa-yy}}{y}+,017453$ Degrees in the Arch whose Co-sine is $\frac{y}{a}$ Radius $=1$.

Ex. 13.

Let AD be the Cycloid, AB=x, BD=y, AE=2r, 105.
FC=u, Arch *AC=v*; by the Nature of the Circle $2ry-yy=uu$, and $\dot{u}=\frac{r\dot{y}-y\dot{y}}{u}$; and $\dot{v}=\frac{r\dot{y}}{u}$: And by the Nature of the Cycloid $x=v+u$, and $\dot{x}=\dot{v}+\dot{u}=\frac{r\dot{y}}{u}+\frac{r\dot{y}-y\dot{y}}{u}=\frac{2r\dot{y}-y\dot{y}}{\sqrt{2ry-yy}}$; whence $\dot{z}=y\dot{x}=\frac{2ry-yy}{\sqrt{2ry-yy}}\dot{y}=\dot{y}\sqrt{2ry-yy}$, but this is the Fluxion of the Area *AFC*, therefore z or *ABD* $=AFC$.

Ex. 14.

Let AD be the Quadratrix, $AB=x$, $BD=y$, then 106.
(by the Procefs in Ex. 11. Prob. VIII.) $\dot{x}=\frac{2y\dot{y}}{3a}+\frac{4y^3\dot{y}}{45a^3}+\frac{4y^5\dot{y}}{315a^5}+$ &c; whence $\dot{z}=y\dot{x}=\frac{2y^2\dot{y}}{3a}$ +

FIG. $+\frac{4y^4\dot{y}}{45a^3}+\frac{4y^6\dot{y}}{315a^5}$ &c; whence $z=\frac{2y^3}{9a}+\frac{4y^5}{225a^3}+\frac{4y^7}{2205a^5}+$ &c.

Ex. 15.

107. *Let AD be the Catenary* $AB=x$, $BD=y$, Curve $AD=v$, the Equation of the Curve is $vv=2ay+yy$, whence $v\dot{v}=a\dot{y}+y\dot{y}$, and $v^2\dot{v}^2=\overline{a+y}^2.\dot{y}^2$, or $\overline{a+y}^2\times\dot{v}^2-a^2\dot{v}^2=\overline{a+y}^2\times\overline{\dot{v}^2-\dot{x}^2}$, whence $a\dot{v}=\overline{a+y}\times\dot{x}$; therefore $\dot{z}=y\dot{x}=a\dot{v}-a\dot{x}$, and $z=av-ax$.

Ex. 16.

108. *Let AD be such a mechanical Curve, that the Ordinate* $FD=$ *Arch* AC *of the Parabola, whose Equation is* $ry=uu$, putting $AB=x$, BD or $AF=y$, $FC=u$, Arch $AC=v$. Then $v=\dot{x}$, and $\dot{x}=\dot{v}=(\sqrt{\dot{y}^2+\dot{u}^2}=)$ $y^{-\frac{1}{2}}\dot{y}\sqrt{\tfrac{1}{4}r+y}$. Then $\dot{z}=y\dot{x}=y\dot{v}=y^{\frac{1}{2}}\dot{y}\sqrt{\tfrac{1}{4}r+y}$. But $v=$ Fluent of $y^{-\frac{1}{2}}\dot{y}\sqrt{\tfrac{1}{4}r+y}$ (by Prob. VIII.) therefore by Form 11th, $z=\overline{\tfrac{1}{8}r+\tfrac{1}{2}y}\sqrt{2ay+yy}-\frac{rv}{16}$.

Ex. 17.

109. *Let AD be a Circle, to find the Area ABD, extremely near the Vertex.* Let Radius $CA=a$, $AB=x$, $BD=y$, then $\dot{z}=y\dot{x}=\dot{x}\sqrt{2ax-xx}$; but in A, $\sqrt{2ax-xx}=\sqrt{2ax}$: therefore (by Cor. 3. Prop. II.) $\dot{z}=y\dot{x}=\dot{x}\sqrt{2ax}$; and the Area $z=\tfrac{2}{3}x\sqrt{ax}=\tfrac{2}{3}xy$, the Area in the very Vertex of the Figure.

COR. After the same Manner in all Curves of a finite Curvature, the Area of the Segment extremely near the Vertex is $\tfrac{2}{3}$ the Base into the Height.

Ex. 18.

Let the Equation of a Curve be $y=\frac{\overline{ax^r+bx^s+dx^t}\big|^m}{\overline{fx^v+gx^w}\big|^n}$, *to find the Area of an extremely small Part of the Figure,* when

when $x=c$. Here $\dot{z}=y\dot{x}=\frac{\overline{ac^r+bc^s+dc^t}|^m}{\overline{fc^v+gc^w}|^n}\times\dot{x}$ FIG.

(by Cor. 3. Prop. II.) therefore $z=\frac{\overline{ac^r+bc^s+dc^t}|^m}{\overline{fc^v+gc^w}|^n}\times\acute{x}$.

Where $\acute{x}$ is a very small Part of the Abscissa.

Cor. Hence the Area of any compound Curve may be nearly found, by finding, after this Manner, all the Parts of the Area belonging to the several small Parts of the Abscissa, and collecting them into one Sum.

Ex. 19.

Let BDm be Archimedes's Spiral, $AB=r$, Arch AC 110.
$=v$, $BD=y$, by the Nature of the Figure $rv=cy$, whence $\dot{v}=\frac{c\dot{y}}{r}$, and by similar Triangles $\dot{v}=\frac{r\dot{x}}{y}$, whence $\dot{x}=\frac{cy\dot{y}}{rr}$; therefore $\dot{z}=\frac{cyy\dot{y}}{2rr}$; and thence the Area BED or $z=\frac{cy^3}{6rr}$.

Ex. 20.

Let QDC be the reciprocal Spiral, Radius $AB=a$, 111.
Arch $AC=b$, Arch $AP=v$, $BD=y$; the Nature of the Curve gives $ab=vy$, whence $v\dot{y}+y\dot{v}=0$, and $\dot{v}=\frac{-v\dot{y}}{y}$, and by similar Triangles $\dot{v}=\frac{a\dot{x}}{y}$, whence $\dot{x}=\frac{-v\dot{y}}{a}$, therefore $\dot{z}=\frac{y\dot{x}}{2}=\frac{-vy\dot{y}}{2a}=\frac{-ab\dot{y}}{2a}=\frac{-b\dot{y}}{2}$, and $z=\frac{-by}{2}$; and when corrected, the Area BCD or $z=b\times\frac{BC-BD}{2}$.

Ex. 21.

Ex. 21.

FIG. 111. *Let QDC be the proportional Spiral*, whose Nature is $b\dot{y} = c\dot{x}$, or $\dot{x} = \frac{b\dot{y}}{c}$, therefore $\dot{z} = \frac{y\dot{x}}{2} = \frac{by\dot{y}}{2c}$: and z or the Area $BQD = \frac{by^2}{4c}$.

Ex. 22.

112. *Let BED be a kind of Spiral*, $BD = y$, Arch AD (whose Center is B) $= v$, and let its Nature be expressed by this Equation $av^2 = y^3$, or $y = a^{\frac{1}{3}}v^{\frac{2}{3}}$, and $\dot{y} = \frac{2}{3}a^{\frac{1}{3}}v^{-\frac{1}{3}}\dot{v}$, whence $\dot{z} = v\dot{y} = \frac{2}{3}a^{\frac{1}{3}}v^{\frac{2}{3}}\dot{v}$, and the Area $z = \frac{2}{5}a^{\frac{1}{3}}v^{\frac{5}{3}} = \frac{2}{5}vy$ for the Area $ABED$.

SCHOLIUM.

It may not be improper, in this Place, to insert (from the Transactions) a general Method for determining the Quadratures of Curves, with it's Investigation.

Let the Equation of a Curve be $y^m + ax^n + by^p x^q + cy^r x^s + dy^t x^u$ &c $= 0$. And it's Area $Ayx + By^l x^f + Cy^g x^h + Dy^i x^k +$ &c $=$ Fluent of $y\dot{x}$. Put these two Equations into Fluxions, and make the two values of $\dot{y}$ equal to each other (expunging y^{m-1} by the Equation of the Curve) and we have $\frac{anx^{n-1} + bqy^p x^{q-1} + csy^r x^{s-1} + \&c}{-max^n y^{-1} \frac{+p}{-m} by^{p-1}x^q \frac{+r}{-m} cy^{r-1}x^s \&c}$

$= \frac{\frac{A}{-1}y + fBy^l x^{f-1} + hCy^g x^{h-1} + \&c}{Ax + lBy^{l-1}x^f + gCy^{g-1}x^h \ \&c}$; and multiplying the Numerators and Denominators alternately, there arises,

$anAx^n$

$$\left.\begin{array}{l} anAx^n + bqAy^p x^q \qquad + csAy^r x^s \\ \qquad + anlBy^{l-1}x^{f+n-1} + lbqBy^{p+l-1}x^{f+q-1} \\ \qquad\qquad\qquad + ganCy^{g-1}x^{h+n-1} \end{array}\right\}\&c =$$

$$\left.\begin{array}{l} \overline{1-A}.max^n + \overline{1-A}\times{}^{\;m}_{-p}by^p x^q \qquad + \overline{1-A}\times{}^{+m}_{-r}cy^r x^s \\ \qquad - mafBy^{l-1}x^{f+n-1} {}^{+p}_{-m}bfBy^{p+l-1}x^{f+q-1} \\ \qquad\qquad\qquad + mahCy^{g-1}x^{h+n-1} \end{array}\right\}\&c.$$

Now for determining the Indices, compare the homologous Terms. Thus $p=l-1$, whence $l=p+1$. Again $r=p+l-1=g-1$, and thence $r=2p$, and $g=2p+1$. After the fame Manner $t=3p$, $i=3p+1$. Put $e+n=q$, and then $f=e+1$, $s=2e+n$, $h=2e+1$, $u=3e+n$, $k=3e+1$, &c. Therefore the Equation of the Curve will be in this Form, $y^m+ax^n+by^p x^{e+n}+cy^{2p}x^{2e+n}+dy^{3p}x^{3e+n}$ &c $=0$, and it's Quadrature $Ayx+By^{p+1}x^{e+1}+Cy^{2p+1}x^{2e+1}+Dy^{3p+1}x^{3e+1}$ &c $=$ Fluent of $y\dot{x}$.

Next for determining the Coefficients A, B, C, &c. Put the Coefficients of the homologous Powers of x equal to each other, thus $anA=\overline{1-A}.ma$, whence $A=\frac{m}{m+n}$. After the fame Manner $B=$

$$\left.\begin{array}{l} m-p \\ +\overline{p-m-e-n}\times A \end{array}\right\}\times\frac{b}{am\times\overline{e+1}+an\times\overline{p+1}},\ C=$$

$$\left.\begin{array}{l} \overline{m-2p}\times c \\ +\overline{2p-2e-m-n}\times cA \\ :+\overline{m-p}\times\overline{e+1}+\overline{e+n}\times\overline{p+1}\times:-bB \end{array}\right\}\times\frac{1}{a\times:m\times\overline{2e+1}+n\times\overline{2p+1}:}$$

&c.

Hence therefore if $y^m+ax^n+by^p x^{e+n}+cy^{2p}x^{2e+n}+dy^{3p}x^{3e+n}$ &c $=0$, *then the Fluent of* $y\dot{x}$, *or the Area of that Curve is,*

$$+\frac{m}{m+n}yx.$$

$$\left.\begin{array}{l} +\overline{m-p}\times b \\ +\overline{p-m-e-n}\times Ab \end{array}\right\}\times\frac{y^{p+1}x^{e+1}}{am\times\overline{e+1}+an\times\overline{p+1}}$$

$$\left.\begin{array}{l} +\overline{m-2p}\times c \\ +\overline{2p-2e-m-n}\times cA \\ +:\overline{m-p}\times\overline{e+1}+\overline{e+n}\times\overline{p+1}:\times-bB \end{array}\right\}\times\frac{y^{2p+1}x^{2e+1}}{am\times\overline{2e+1}+an\times\overline{2p+1}}$$

$$\left.\begin{array}{l} +\overline{m-3p}\times d \\ +\overline{3p-3e-m-n}\times dA \\ +:\overline{m-2p}\times\overline{e+1}+\overline{2e+n}\times\overline{p+1}:\times-cB \\ +:\overline{m-p}\times\overline{2e+1}+\overline{e+n}\times\overline{2p+1}:\times-bC \end{array}\right\}\times\frac{y^{3p+1}x^{3e+1}}{am\times\overline{3e+1}+an\times\overline{3p+1}}$$

$$\left.\begin{array}{l} +\overline{m-4p}\times e \\ +:\overline{4p-4e-m-n}:\times eA \\ +:\overline{m-3p}\times\overline{e+1}+\overline{3e+n}\times\overline{p+1}:\times-dB \\ +:\overline{m-2p}\times\overline{2e+1}+\overline{2e+n}\times\overline{2p+1}:\times-cC \\ +:\overline{m-p}\times\overline{3e+1}+\overline{e+n}\times\overline{3p+1}:\times-bD \end{array}\right]\times\frac{y^{4p+1}x^{4e+1}}{ma\times\overline{4e+1}+na\times\overline{4p+1}}$$

1. Here note, *A*, *B*, *C*, *D*, *&c*, repreſent the firſt, ſecond, third, fourth *&c* Terms, only leaving out the Powers of x and y.

2. This Series ſometimes gives the Quadrature of a Curve in finite Terms; particularly in Trinomials (that is whoſe Equation conſiſts of three Terms) when $\frac{e-p+m+n}{-em-pn}$ is a poſitive whole Number; to which add 1, and you have the Number of Terms in the Series.

3. But in Curves of more Terms, there are ſeveral Conditions requiſite to their exact Quadrability, which it is needleſs to enumerate, becauſe ſuch Curves ſeldom admit of an exact Quadrature. It is ſufficient to obſerve, that if $N=$ Number of Terms in the Equation of the Curve, they will ſometimes admit of ſuch a Quadrature when $\frac{Ne-2e+2p-Np+m+n}{-em-pn}+1$ is a poſitive whole Number, but never elſe; which Num-

ber

ber will then shew the Number of Terms in the Series, that constitutes the Area.

4. When the Quadrature of a Curve is required by this Series; reduce the Equation of the Curve to the preceding Form, and comparing the homologous Terms, the Exponents and Coefficients will be easily determined; which must be substituted into the foregoing general Series, as usual. And when any particular Terms are wanting in the given Equation, then the respective Coefficients will be o, and those Terms of the Series wherein they are found will vanish.

5. And we must first of all enquire, whether it will admit of an exact or geometrical Quadrature, and if it will not admit of it in one Form, it may in another. To this Purpose we must divide the whole Equation by some Powers of x and y; so that in the resulting Equation, there may always be one Term without x and another without y; for this Condition is absolutely necessary to the Equation; and thus you will have a new Form: And this we must do as often as possible. Now the Number of different Forms any Equation will admit of is $nn-n$, putting $N=$ Number of Terms in the Equation.

6. If it admit of such Quadrature in none of these Forms, try to find the Complement of the Area, by writing x for y and y for x in the Equation of the Curve; and then proceeding with the new Equation in all Respects as before.

Example 1.

Let $y^3+x^3-byx=0$. Here $m=3$, $n=3$, $p=1$, $e=-2$, $a=1$, $b=-b$, and $\frac{e-p+m+n}{-em-pn}=1$; whence the Curve is geometrically quadrable, and the Area $=\frac{1}{2}xy-\frac{y^2b}{6x}$.

Ex. 2.

Suppose $y^{\frac{5}{3}}-ax^{\frac{1}{4}}+ry^7x^{\frac{3}{2}}=0$. Here $m=\frac{5}{3}$, $n=\frac{1}{4}$, $a=$

FIG. $a=-a$, $b=r$, $p=7$, $e=\frac{5}{4}$: then $\frac{e-p+m+n}{-em-pn}=1$; and the Area $=\frac{20}{23}xy+\frac{8r}{23a}y^8x^{\frac{9}{4}}$.

Ex. 3.

Let $y^2-\frac{4g^2}{b}x^6+\frac{g}{b}y^2x^4=0$; here the Curve is not quadrable in this Form, therefore divide by y^2, and then $1+\frac{g}{b}x^4-\frac{4gg}{b}y^{-2}x^6=0$, where $m=0$, $n=4$, $p=-2$, $e=2$, $a=\frac{g}{b}$, $b=-\frac{4gg}{b}$, and $\frac{e-p+m+n}{-em-pn}=1$; whence the Area $=\frac{2gx^3}{y}$.

Ex. 4.

Let $y^2-ax^{-1}-2ay^3x-ay^6x^3=0$, here $m=2$, $n=-1$, $a=-a$, $b=-2a$, $c=-a$, $p=3$, $e=2$, whence $\frac{2e-2p+m+n}{-em-pn}=1$; and the Area of the Figure is $=2xy-\frac{2}{5}x^3y^4$: For all the following Terms of the Series will be nothing.

PROB. XI.

To find any Number of Curves that may be squared.

113. *Let the Abscissa AB$=x$, Ordinate BD$=y$, Area ABD$=z$. Assume any Equation between x and z, this will determine the Area: From that Equation get $\dot{z}$, and substitute it's value in the Equation $\dot{z}=y\dot{x}$, and this will give the Nature of the Curve.*

Ex. 1. *Let* $x^2=z$, whence $2x\dot{x}=\dot{z}=y\dot{x}$, whence $y=2x$, and the Figure is a Triangle.

Ex. 2.

Ex. 2. *Let* $ax^3=z^2$, and $\dot{z}=\frac{3}{2}a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x}=y\dot{x}$, whence $\frac{3}{2}\sqrt{ax}=y$, an Equation to a Parabola.

Likewise if $x^3=az$, then $\frac{3x^2\dot{x}}{a}=\dot{z}=y\dot{x}$; whence $3x^2=ay$, an Equation again to the Parabola.

Ex. 3. *Let* $\frac{a^6}{xx}=z^2$, whence $-\frac{a^3}{xx}\dot{x}=\dot{z}=y\dot{x}$; therefore $-\frac{a^3}{xx}=y$, whence y being negative lyes on the other Side of *AB*.

Ex. 4. *Let* $ccaa+ccxx=zz$, or $z=c\sqrt{aa+xx}$, then $\dot{z}=\frac{cx\dot{x}}{\sqrt{aa+xx}}=y\dot{x}$; therefore $y=\frac{cx}{\sqrt{aa+xx}}$.

Ex. 5. *If* $\frac{\overline{aa+xx}|^{\frac{3}{2}}}{b}=z$, then $\dot{z}=\frac{3x\dot{x}}{b}\sqrt{aa+xx}$ $=y\dot{x}$, and therefore $y=\frac{3x}{b}\sqrt{aa+xx}$.

Ex. 6. Assume $b-3xz+\frac{2}{3}z=z^2$, then $\frac{2}{3}\dot{z}-3x\dot{z}-3z\dot{x}=2z\dot{z}$, and $\dot{z}=\frac{3z\dot{x}}{\frac{2}{3}-3x-2z}=y\dot{x}$; therefore $y=\frac{3z}{\frac{2}{3}-3x-2z}$, out of which z may be expunged by Help of the assumed Equation.

Ex. 7. *Let* $\overline{e+fx^n}|^m=z$, then $\dot{z}=mnfx^{n-1}\dot{x}\times\overline{e+fx^n}|^{m-1}=y\dot{x}$; whence $y=mnfx^{n-1}\times\overline{e+fx^n}|^{m-1}$.

PROB.

PROB. XII.

FIG. *Any Curve ABD being given; to find any Number of Curves ACE whose Areas shall have any assigned relation to the Area of the given Curve.*

114. Let ADF be any Curve referred to an Axis, or
115. ADF any mechanical Curve, whose Complement is
116. ABD; or ADH any Spiral described by the Arch
117. BD, whose Center is A. ACE the Curve required, whose Relation to the first is given.

Put $AB=x$, $BD=y$, Area $ABD=z$.
$AC=v$, $CE=u$, Area $ACE=w$.

Assume any two Equations, one of which may contain the Relation of the Areas z, w; *the other the Relation of the Abscissa or Ordinate of the given Curve AD* (x *or* y) *to the Abscissa or Ordinate* (v *or* u) *of the other Curve AE. By Help of these two Equations, and this third* $y\dot{x}=\dot{z}$, *expunge all Quantities as far as possible, except* v *and* u, *out of the Equation* $u\dot{v}=\dot{w}$, *and you will get an Equation for the Nature of the Curve ACE: And the second assumed Equation will determine the Quantities of the Abscissas or Ordinates of the two Curves to have the required Relation.*

114. Ex. 1. Let AD be a Circle whose Equation is
117. $ax-xx=yy$. Assume $ax=vv$, and $z=w$, then $\dot{x}=\frac{2v\dot{v}}{a}$; whence $u\dot{v}=\dot{w}=\dot{z}=y\dot{x}=\frac{2v\dot{v}}{a}\sqrt{ax-xx}=\frac{2v\dot{v}}{a}\sqrt{vv-\frac{v^4}{aa}}$: therefore $u=\frac{2vv}{aa}\sqrt{aa-vv}$, the Equation of the Curve, whose Area is equal to that of the Circle, when $ax=vv$.

Ex. 2.

Ex. 2. Let $ax - xx = yy$ as before, and $ax + z = v$, and $z = w$. Then $u\dot{v} = \dot{w} = \dot{z} = y\dot{x} = \frac{\dot{v} - \dot{z}}{a}y = \frac{\dot{v} - u\dot{v}}{a}y$: Whence $u = \frac{y - uy}{a}$, or $u = \frac{y}{a+y}$, an Equation to a mechanical Curve. FIG. 114. 117.

Ex. 3. Again let $ax - xx = yy$, $cx + z = w$, and $ax = vv$. Then $u\dot{v} = \dot{w} = c\dot{x} + \dot{z} = c\dot{x} + y\dot{x} = \frac{2cv\dot{v}}{a} + \frac{2v\dot{v}}{a}\sqrt{v^2 - \frac{v^4}{aa}}$. Whence $u = \frac{2vv}{aa}\sqrt{aa - vv} + \frac{2cv}{a}$.

Ex. 4. Let $ax - xx = yy$ as before, and assume $z - \frac{2y^3}{3a} = w$, and $x = v$. Then $u\dot{v} = \dot{w} = \dot{z} - \frac{2y^2\dot{y}}{a} = y\dot{x} - \frac{2y^2\dot{y}}{a} = \frac{2x\dot{x}}{a} \times \sqrt{ax - xx} = \frac{2v\dot{v}}{a}\sqrt{av - vv}$; hence $u = \frac{2v}{a}\sqrt{av - vv}$.

Ex. 5. Again let $ax - xx = yy$, and $z^2 = w$, $x = v^2$. Then $u\dot{v} = \dot{w} = 2z\dot{z} = 2zy\dot{x} = 4zyv\dot{v} = 4zv\dot{v}\sqrt{ax - xx} = 4zv\dot{v}\sqrt{av^2 - v^4} = 4zv^2\dot{v}\sqrt{a - v^2}$: Hence $u = 4zvv\sqrt{a - vv}$, an Equation to a mechanical Curve.

Ex. 6. Let $cc + xx = yy$ an Equation to an Hyperbola; and assume $z = w$, $xx = cv$. Then $u\dot{v} = \dot{w} = \dot{z} = y\dot{x} = \frac{c\dot{v}}{2\sqrt{cv}} \times \sqrt{cc + cv} = \frac{c\dot{v}}{2v}\sqrt{cv + vv}$. Hence $u = \frac{c}{2v}\sqrt{cv + vv}$. 98. 117.

Ex. 7. Let $cc + xx = yy$ as before, and $xy - z = w$, $xx = cv$. Then $u\dot{v} = \dot{w} = x\dot{y} + y\dot{x} - \dot{z} = x\dot{y} = \frac{x^2\dot{x}}{\sqrt{cc + xx}} = \frac{cv\dot{v}}{2\sqrt{cv + vv}}$. Hence $u = \frac{cv}{2\sqrt{cv + vv}}$.

Ex. 8. Let $y = \frac{xx}{\sqrt{ax - xx}}$ an Equation to the Cissoid; and $\frac{2x}{3}\sqrt{ax - xx} + \frac{1}{3}z = w$, and $x = v$. Then 103. 117.

FIG. Then $u\dot{v}=\dot{w}=\frac{2}{3}\dot{x}\times\sqrt{ax-xx}+\frac{2ax\dot{x}-4x^2\dot{x}}{6\sqrt{ax-xx}}+\frac{1}{3}\dot{z}$ $=\frac{ax-xx}{\sqrt{ax-xx}}\dot{x}=\dot{x}\sqrt{ax-xx}=\dot{v}\sqrt{av-vv}$. Hence $u=\sqrt{av-vv}$, an Equation to the Circle.

114. Ex. 9. Let $y=dx^r\times\overline{e+fx^n}|^m$; assume $z=w$, $v=$
117. $\overline{e+fx^n}|^m$, then will $x=\overline{\frac{v^{\frac{1}{m}}-e}{f}}|^{\frac{1}{n}}$, and $\dot{x}=\frac{v^{\frac{1}{m}-1}\dot{v}}{mnf^{\frac{1}{n}}}\times$ $\overline{v^{\frac{1}{m}}-e}|^{\frac{1}{n}-1}$, $y=dv\times\overline{\frac{v^{\frac{1}{n}}-e}{f}}|^{\frac{r}{n}}$; whence $u\dot{v}=\dot{w}=$ $\dot{z}=y\dot{x}=\frac{dv^{\frac{1}{m}}\dot{v}}{mnf^{\frac{r+1}{n}}}\times\overline{v^{\frac{1}{m}}-e}|^{\frac{r+1}{n}-1}$: therefore $u=$ $\frac{dv^{\frac{1}{m}}}{mnf^{\frac{r+1}{n}}}\times\overline{v^{\frac{1}{m}}-e}|^{\frac{r+1}{n}-1}$.

118. Ex. 10. Let BD or $y=$ Arch AR of the Parabo-
117. la AR, whose Equation $ax=ss$, putting $AR=p$, $BR=s$, Tangent $TR=t$; then $\dot{s}=\frac{s\dot{p}}{t}$. Now assume $z=w$, $y=v$, then will $u\dot{v}=\dot{w}=\dot{z}=y\dot{x}=$ $\frac{2vs\dot{s}}{a}=\frac{2vss\dot{p}}{at}=\frac{2vx}{t}\dot{p}=\frac{2vx}{t}\dot{y}=\frac{2vx}{t}\dot{v}$; consequently $u=\frac{2vx}{t}$, a mechanical Curve.

115. Ex. 11. Let AD be a Cycloid, Diameter of the
117. generating Circle $=a$, $FG=s$. Assume $y=v$, $z=w$, then by the Property of the Cycloid $y\dot{x}=s\dot{y}$: whence $u\dot{v}=\dot{w}=\dot{z}=y\dot{x}=s\dot{y}=s\dot{v}$; and therefore $u=s$; consequently AE is a Circle the same with AG.

Ex. 12. Let AD be a Figure of Arches, where Arch $AG=AB=x$. Assume $y=v$, $z=w$; and by the Property of the Circle $\dot{x}=\frac{a\dot{y}}{2\sqrt{ay-yy}}$; then

then $u\dot{v} = \dot{w} = \dot{z} = y\dot{x} = \frac{av\dot{y}}{2\sqrt{ay-yy}} = \frac{av\dot{v}}{2\sqrt{av-vv}}$. FIG.

Hence $u = \frac{av}{2\sqrt{av-vv}}$.

Ex. 13. Let AD be the Spiral of Archimedes, whose Equation is $by = xx$: And assume $x = v$, and $z = w$; then $u\dot{v} = \dot{w} = \dot{z} = y\dot{x} = y\dot{v} = \frac{xx}{b}\dot{v} = \frac{vv}{b}\dot{v}$; whence $bu = vv$, and AE a Parabola convex towards AC. 116. 117.

PROB. XIII.

To find the Surface of a Solid.

As the Fluxion of any Space is equal to the describing Line drawn into the Fluxion of the Axis; so the Fluxion of the Surface of a Solid (generated by a Line revolving about an Axis) is equal to the Periphery of that Circle drawn into the Fluxion of the Line generating that Surface. Therefore

Let the Abscissa $AP = x$, Ordinate $PM = y$, Curve AM (or BM) $= z$, $c = 3.1416 \times 2 =$ the Circumference of the Circle whose Radius is 1, then to find the Surface s generated by the Curve AM revolving about the Axis AP; by the Equation of the Curve expunge one of the indeterminea Quantities and it's Fluxion out of the Equation $\dot{s} = cy\dot{z}$, or $cy\sqrt{\dot{x}^2 + \dot{y}^2}$; and find the Fluent. 119. 120. 121.

Ex. 1.

Let ABD be a right Cone, $AB = a$, $BC = b$, then $y = \frac{bz}{a}$, therefore $\dot{s} = cy\dot{z} = \frac{cbz\dot{z}}{a}$, and $s = \frac{cbz^2}{2a}$ 123.

 =

FIG. $= \frac{czy}{2}$. And the Surface of the whole Cone *ABD* $= \frac{c}{2} \times AB \times BC$.

Ex. 2.

119. *Let AM be a spherical Surface*, Radius $= a$, by the Property of the Circle $a\dot{x} = y\dot{z}$; therefore $\dot{s} = cy\dot{z} = ca\dot{x}$, whence $s = cax$.

Ex. 3.

120. *Let AM be a parabolic Surface*, $ay = xx$, then $\dot{y} = \frac{2x\dot{x}}{a}$, whence $\dot{s} = cy\sqrt{\dot{x}^2 + \dot{y}^2} = \frac{cx^2\dot{x}}{aa}\sqrt{aa + 4xx}$: and the Fluent (corrected) is $s = \frac{caax + 8cx^3}{32aa}\sqrt{aa + 4xx} - \frac{ca^2}{64} \times 2.302585 \text{ Log.} \frac{2x + \sqrt{aa + 4xx}}{a}$, found by Forms the 9th, 11th, and 13th.

Ex. 4.

119. *Let $ax^m = y^{m+1}$ be an Equation to infinite Parabolas.* By the Process in Ex. 4. Prob. VIII. $\dot{z} = \dot{y}\sqrt{1 + by^{\frac{2}{m}}}$; whence $\dot{s} = cy\dot{y}\sqrt{1 + by^{\frac{2}{m}}} = cy^{\frac{1}{m}+1}\dot{y}\sqrt{b + y^{-\frac{2}{m}}}$. Therefore the Fluent of $\dot{s}$ or $cy\dot{y}\sqrt{1 + by^{\frac{2}{m}}}$ will be had by Form the 15th, when m is the half of any negative odd Number: And the Fluent of $cy^{\frac{1}{m}+1}\dot{y}\sqrt{b + y^{-\frac{2}{m}}}$ will be had also by Form the 15th when m is any positive whole Number. And likewise the Fluent of $cy\dot{y}\sqrt{1 + by^{\frac{2}{m}}}$ will be had by Form the 11th when m is the half of an odd Number; first finding the Fluent of $cy^{\frac{1}{m}-1}\dot{y}\sqrt{1 + by^{\frac{2}{m}}}$, by Form the 9th and 13th. In other Cases, Form the 15th or 16th will give the Fluent by infinite Series.

COR. If $m = 1$, then $s = \frac{c}{12a} \times \overline{aa + 4yy}|^{\frac{3}{2}} - \frac{caa}{12}$.

Ex. 5.

Ex. 5.

In the elliptic Surface BM, whose Center is A, let $AD = a$, $AB = b$, $AP = x$, $PM = y$; then $y = \sqrt{bb - \frac{bb}{aa}xx}$, and $\dot{y} = \frac{-bbx\dot{x}}{a\sqrt{aabb - bbxx}}$, and $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2} = \frac{b\dot{x}}{aay}\sqrt{a^4 - aaxx + bbxx} = \frac{b\dot{x}}{aay}\sqrt{a^4 \mp ddxx}$; putting $aa \backsim bb = dd$. Therefore $\dot{s} = cy\dot{z} = \frac{cb\dot{x}}{aa}\sqrt{a^4 \mp ddxx}$. Let $Q = ,017453 \times$ Degrees in the Arch whose Sine is $\frac{dx}{aa}$, when a is greater than b. Or $Q = 2.30258$ Log. $\frac{dx + \sqrt{a^4 + ddxx}}{aa}$, when a is less than b. And we shall have (by Form 10th and 13th, or by Form the 9th and 13th) $s = \frac{cbaa}{2d}Q + \frac{cbx}{2aa}\sqrt{a^4 + \overline{bb - aa} \cdot x^2}$, for the Surface BM revolving round AP. FIG. 121.

Ex. 6.

In the Hyperboloid BM, described by revolving about AP, let the Semi-conjugæ $= b$, Semi-transverse $AB = a$, $AP = x$, $PM = y$; then $y = \sqrt{\frac{bbxx}{aa} - bb}$, $\dot{y} = \frac{bx\dot{x}}{a\sqrt{xx - aa}}$, and $\sqrt{\dot{x}^2 + \dot{y}^2} = \frac{b\dot{x}}{aay}\sqrt{ddxx - a^4}$, putting $dd = aa + bb$. Therefore $\dot{s} = cy\sqrt{\dot{x}^2 + \dot{y}^2} = \frac{cb\dot{x}}{aa}\sqrt{ddxx - a^4}$; whence (by Form the 9th and 13th) $s = \frac{cbx}{2aa}\sqrt{ddxx - a^4} - \frac{cbaa}{2d} \times 2.30258$ Log. $dx + \sqrt{ddxx - a^4}$: And the Fluent corrected, $s = \frac{cbx}{2aa}\sqrt{ddxx - a^4} - \frac{cbb}{2} + \frac{cbaa}{2d} \times 2.3025$ Log. $\frac{da + ba}{dx + \sqrt{ddx^2 - a^4}}$. 124.

Ex. 7.

FIG. 124. *To find the Surface described by revolving round the Conjugate AC.* Here $x = \frac{a}{b}\sqrt{bb+yy}$, and $\dot{x} = \frac{a y \dot{y}}{b\sqrt{bb+yy}}$, and $\sqrt{\dot{x}^2+\dot{y}^2} = \dot{y}\sqrt{\frac{bb+\frac{dd}{bb}yy}{bb+yy}}$. The Circumference described by M is cx, whence $\dot{s} = cx\dot{z}$ $= \frac{ca}{b}\dot{y}\sqrt{bb+\frac{dd}{bb}yy}$. And (by Forms the 9th and 13th) $s = \frac{cay}{2b}\sqrt{bb+\frac{dd}{bb}yy} + \frac{2.302585cabb}{2d} \times$ Log. $\frac{dy}{bb} + \sqrt{1+\frac{ddyy}{b^4}}$.

Ex. 8.

122. *In the right angled Hyperbola CM,* let $AP = x$, $PM = y$, and $aa = xy$, and $\dot{y} = \frac{-y\dot{x}}{x} = \frac{-aa\dot{x}}{xx}$, whence $\sqrt{\dot{x}^2+\dot{y}^2} = \frac{\dot{x}}{xx}\sqrt{a^4+x^4}$, and therefore $\dot{s} = cy\sqrt{\dot{x}^2+\dot{y}^2} = \frac{caa\dot{x}}{x^3}\sqrt{a^4+x^4} = \frac{ca^6x^{-5}\dot{x}}{\sqrt{1+a^4x^{-4}}} + \frac{caax\dot{x}}{\sqrt{a^4+x^4}}$. Whence (by Forms the 3d and 9th) $s = \frac{caa}{2} \times 2.30258 \times$ Log. $x^2+\sqrt{a^4+x^4} : -\frac{caa}{2xx}\sqrt{a^4+x^4}$. But in C, let $x = a$, and $s = 0$, therefore the Fluent corrected is $s = \frac{caa}{\sqrt{2}} - \frac{caa}{2xx}\sqrt{a^4+x^4} + \frac{caa}{2} \times$ 2.30258 Log. $\frac{x^2+\sqrt{a^4+x^4}}{aa \times 1+\sqrt{2}}$, for the Surface described by *CM* about the Assymptote *AP*.

Ex. 9.

Let the Ellipsis Mm revolve round the Line AP perpendicular to the Axis CA. Let $CA = d$, $RM = Qm = z$, then $\dot{s} = c\dot{z} \times PM + c\dot{z} \times Pm = 2cd\dot{z}$: Whence $s = 2cdz$. And the whole Surface described by $RDQ = cd \times$ Semi-periphery RDQ. FIG. 125.

Ex. 10.

To find the Surface of the Ungula of a Cylinder cut off by a Plane FQ. Let F be the Vertex, ADZ the Section made by a Circle perpendicular to the Axis; FD, NI Perpendiculars on the Plane of the Circle ADZ; DC, IP Perpendiculars on AZ. 126.

Put $CD = d$, $DF = a$, Radius of the Cylinder $= r$, $DI = z$, $CP = x$, then $r - d + PI =$ Cosine of $z = y$, or $PI = y + d - r$; and by the Nature of the Circle $r\dot{x} = y\dot{z}$, and by similar Triangles $NI = \frac{a}{d} \times \overline{y + d - r}$; but $\dot{s} = NI \times \dot{z} = \frac{a\dot{z}}{d} \times \overline{y + d - r} = \frac{ar\dot{x}}{d} + \frac{a\dot{z}}{d} \times \overline{d - r}$: Whence $s = \frac{arx}{d} + \frac{az}{d} \times \overline{d - r}$; and the whole Surface $ZFADZ = a \times ADZ - \frac{ar}{d} \times \overline{\text{Arch } ZDA - \text{Cord } ZA}$.

Ex. 11.

Let the Cissoid NM revolve round the Assymptote AP; let $AN = a$, $AP = x$, $PM = y$, then is $x = \frac{\overline{a - y}|^2}{\sqrt{ay - yy}} = \frac{\overline{a - y}|^{\frac{3}{2}}}{\sqrt{y}}$; whence $\dot{x} = \frac{-a\dot{y} - 2y\dot{y}}{2yy} \sqrt{ay - yy}$; therefore $\dot{z} = \frac{a\dot{y}}{2y} \sqrt{\frac{a + 3y}{y}}$; consequently $\dot{s} = cy\dot{z} = \frac{ca\dot{y}}{2} \sqrt{\frac{a + 3y}{y}} = \frac{cay^{-\frac{1}{2}}\dot{y}}{2} \sqrt{a + 3y}$: From whence (by Forms the 9th and 13th) $s =$ 127.

$\frac{1}{2}ca\sqrt{ay + 3yy}$

FIG. $\frac{1}{2}ca\sqrt{ay+3yy}+\frac{caa}{4}\times 2.3025\,\text{Log.}\,a+6y+2\sqrt{3ay+9y^2}$: and the Fluent being duly corrected, $s=\frac{1}{2}ca\sqrt{ay+3yy}-caa+\frac{2.302585caa}{4}\times\text{Log.}\frac{a+6y+2\sqrt{3ay+9yy}}{a\times\overline{7+4\sqrt{3}}}$.

Ex. 12.

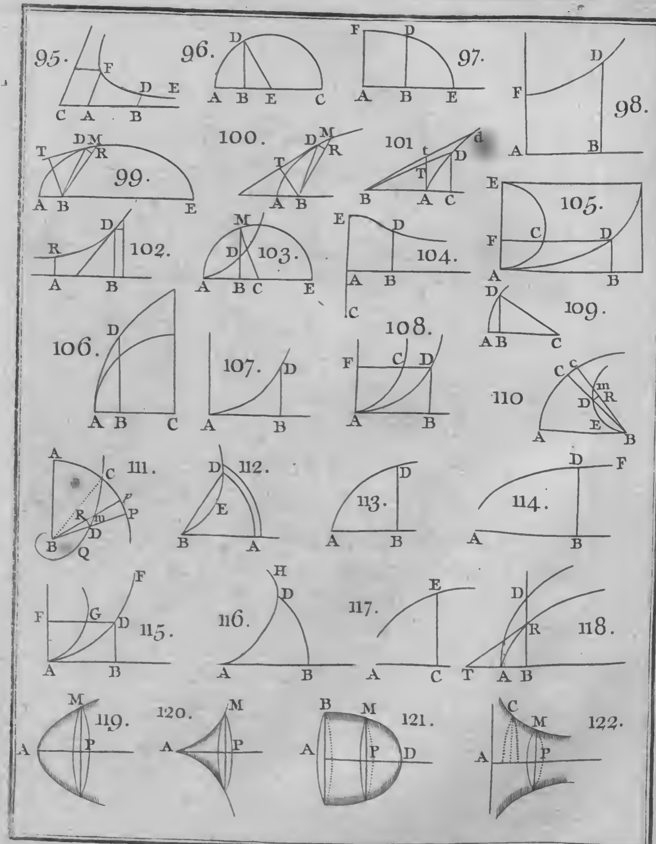
128. *Let DM be the Log. Curve; to find the Surface it describes by revolving round the Assymptote AP.* Put Subtangent $PT=a$, $AD=b$, $AP=x$, $PM=y$. By the Nature of the Curve $\dot{x}=\frac{a\dot{y}}{y}$: Whence $\dot{s}=cy\sqrt{\frac{aa\dot{y}^2}{y2}+\dot{y}^2}=c\dot{y}\sqrt{aa+yy}$; and (by Form 9 and 13) $s=\frac{1}{2}y\sqrt{aa+yy}+\frac{1}{2}caa\times 2.3025\,\text{Log.}\,y+\sqrt{aa+yy}$: but in D, $s=0$, and $y=b$; therefore the Fluent corrected by Prop. XII. is $s=\frac{1}{2}y\sqrt{aa+yy}-\frac{1}{2}b\sqrt{aa+bb}+\frac{1}{2}caa\times 2.302585\,\text{Log.}\frac{y+\sqrt{aa+yy}}{b+\sqrt{aa+bb}}$.

Ex. 13.

129. *Let the Cycloid AM revolve about RC*: Let $AP=x$, $PM=y$, Curve $AM=z$. By the Process Ex. 8. Prob. VIII. $\dot{z}=\dot{x}\sqrt{\frac{a}{x}}$, therefore $\dot{s}=c\times PR\times\dot{x}\sqrt{\frac{a}{x}}=\overline{a-x}\times c\dot{x}\sqrt{\frac{a}{x}}=ca^{\frac{3}{2}}x^{-\frac{1}{2}}\dot{x}-ca^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x}$, whence $s=2ca^{\frac{3}{2}}x^{\frac{1}{2}}-\frac{2}{3}ca^{\frac{1}{2}}x^{\frac{3}{2}}=c\sqrt{ax}\times\overline{2a-\frac{2}{3}x}$. And when $x=a$, the whole Surface described by $AMC=\frac{4}{3}caa$.

Ex. 14.

119. *Suppose the Catenary AM to revolve round the Axis AP*, by Nature of the Curve $zz=2ax+xx$, or $a+x=\sqrt{aa+zz}$, whence $\dot{x}=\frac{z\dot{z}}{\sqrt{aa+zz}}$; likewise $\dot{y}=$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p 206 V.p.

$\dot{y}=\sqrt{\dot{z}^2-\dot{x}^2}=\frac{a\dot{z}}{\sqrt{aa+zz}}$. Since $\dot{s}=cy\dot{z}$, assume FIG.
the Fluent $s=cyz-t$ (by Rule 8. Prop. X.) This put into Fluxions is $\dot{s}=cy\dot{z}+cz\dot{y}-\dot{t}$, or $\dot{t}=cz\dot{y}=\frac{caz\dot{z}}{\sqrt{aa+zz}}$, and (by Form the 3d) $t=ca\sqrt{aa+zz}$; whence $s=cyz-ca\sqrt{aa+zz}=cyz-caa-cax$. And when corrected $s=cyz-cax$.

PROB. XIV.

To find the Content of solid Bodies.

In any Solid *AMm*, generated by the Space *APM* revolving round the Axis *AP*; suppose the Plane *Mm* to move along the Axis *AP*, and by that Motion to describe or generate that Solid; and suppose a given Rectangle *B* to move with the same Motion along the same Axis *AP*, and by that Motion to generate a Parallelopipedon; the Fluxions of these Solids will be as the describing Planes *Mn* and *B* drawn into the Velocities of their Motions or the Fluxions of the Abscissa *AP*. Now since $B\times AP=$ Parallelopipedon, and $B\times$ Fluxion of $AP=$ it's Fluxion, consequently the Fluxion of the Solid $AMm=$ describing Plane $Mm\times$ into the Fluxion of *AP*. 130.

Likewise if the Solid *AMb* be supposed to be generated by a cylindric Surface *MHbm* continually expanding it self and moving along the Ordinate *PM*, still retaining *AP* for it's Axis; it may be the same way proved, that the Fluxion of the Solid *AMHbmA* or *HMEmH* generated thereby is $=$ cylindric Surface *MHbm* multiplied by the Fluxion of the Ordinate *PM*. Therefore, 131.

To

FIG. *To find the Solidity of a Body, let the Abſciſſa $AP=x$, Ordinate $PM=y$, $MH=u$, $c=3.1416$ &c $s=$ ſolid Content. Then by the Equation of the Curve expunge one of the indetermined Quantities (and it's Fluxion) out of the Equation $\dot{s}=cyy\dot{x}$ for the Solid AMm, or out of the Equation $\dot{s}=2cuy\dot{y}$ for the cylindrical Solid AMHbm or HMEmb; and then find the Fluent.*

Example 1.

123. *Let ABD be a Cone,* Height $AC=a$, CB or $CD=b$, then $bx=ay$; therefore $\dot{s}=cyy\dot{x}=\frac{cbbx^2\dot{x}}{aa}$; and $s=\frac{cb^2x^3}{3aa}=\frac{cyyx}{3}$, and the whole Cone $=\frac{cabb}{3}=$ Baſe $\times\frac{1}{3}$ Height.

Ex. 2.

132. *Let BF be a Priſmoid, whoſe Baſes are right angled Parallelograms, though not ſimilar.*

Let it's perpendicular Height $AL=h$, $AP=x$, $AD=d$, $BE=s$, $AG=n$, $BC=m$, by ſimilar Triangles $h:x::s-d:\frac{s-d}{h}x=QI$, and $h:x::m-n:\frac{m-n}{h}x=IR$; then $KI=d+\frac{s-d}{h}x$, and $IH=n+\frac{m-n}{h}x$; then $\dot{s}=\overline{d+\frac{s-d}{h}x}\times\overline{n+\frac{m-n}{h}x}\times\dot{x}=dn\dot{x}+\frac{m-n}{h}dx\dot{x}+\frac{s-d}{h}nx\dot{x}+\overline{s-d}\times\frac{m-n}{hh}x^2\dot{x}$; therefore $s=dnx+\frac{m-n}{2h}dx^2+\frac{s-d}{2h}nx^2+\frac{\overline{s-d}\times\overline{m-n}}{3hh}x^3$ for the Solid *HIKF*. And when $x=h$, then the whole Solid $BF=dm+sn+2sm+2dn\times\frac{h}{6}=\overline{s+\frac{1}{2}d}\times m+\overline{d+\frac{1}{2}s}\times n\times\frac{1}{3}h$.

Ex. 3

Ex. 3.

Let AMm be the Segment of a Sphere; it's Diameter $= a$; then $yy = ax - xx$, therefore $\dot{s} = cyy\dot{x} = cax\dot{x} - cx^2\dot{x}$. Therefore $s = \frac{cax^2}{2} - \frac{cx^3}{3}$, for the Segment $AMPm$. And when $x=a$, the whole Sphere $= \frac{ca^3}{6} = \frac{2}{3}$ the circumscribing Cylinder. FIG. 130.

Ex. 4.

Let AMm be the Segment of a Sphere of an exceeding small Height, here $yy=ax$ nearly, therefore (by Cor. 3. Prop. II.) $\dot{s} = cyy\dot{x} = cax\dot{x}$; therefore $s = \frac{caxx}{2} = \frac{cxy}{2}$, nearly.

Hence in any Solid of a finite Curvature, the Content of a Segment of a very small Height, will be found to be half the Base drawn into the Height.

Ex. 5.

Let the Solid be a Paraboloid, where $ax=yy$; then $\dot{s} = cyy\dot{x} = cax\dot{x}$; therefore $s = \frac{caxx}{2} = \frac{cyyx}{2} = \frac{1}{2}$ Base $\times$ Height. 130.

And in general if $x=y^m$, then $\dot{x} = my^{m-1}\dot{y}$; and then $\dot{s} = cyy\dot{x} = cmy^{m+1}\dot{y}$; and therefore $s = \frac{cm}{m+2}y^{m+2} = \frac{cm}{m+2}y^2x$. Therefore in the conic Parabola, where $m = \frac{1}{2}$, that is when the Curve is convex towards AP, then the Solidity $= \frac{cyyx}{5} = \frac{1}{5}$ Base $\times$ Height.

Ex. 6.

Let AmD be a parabolic Spindle, generated by the Parabola AMD (whose Axis is DT) revolving round the Ordinate AT. 130.

Let $AT=b$, $DT=d$, $DH=v$, $MH=u$, then MP or $y=d-v$, and $av=uu$ by the Nature of the Curve: Whence

FIG. Whence $\dot{y} = -\dot{v} = \frac{-2u\dot{u}}{a}$; wherefore $\dot{s} = 2cuy\dot{y} = \frac{-4cu^2y\dot{u}}{a} = \frac{-4cu^2\dot{u}}{a} \times \overline{d - \frac{uu}{a}}$. Therefore the Fluent $s = \frac{-4cdu^3}{3a} + \frac{4cu^5}{5aa}$: but in *A* where s is o, $u = b$; therefore the Fluent corrected is $s = \frac{4cdb^3}{3a} - \frac{4cdu^3}{3a} + \frac{4cu^5}{5aa} - \frac{4cb^5}{5aa} = \frac{8cdb^3}{15a} - \frac{4cu^3}{15a} \times \overline{v-d}$, for the Solid *AMHbm*; and when v and $u = o$, the whole Solid $ADd = \frac{8cdb^3}{15a} = \frac{8cbdd}{15} = \frac{8}{15}$ the Base $\times$ Height.

Ex. 7.

130. *Let AMm be an Hyperboloid,* Transverse $= a$, Conjugate $= b$. Then $yy = \frac{bb}{aa} \times \overline{ax + xx}$; and $\dot{s} = cyy\dot{x} = \frac{cbb\dot{x}}{aa} \times \overline{ax + xx}$. And the Fluent $s = \frac{cbbxx}{2a} + \frac{cbbx^3}{3aa} =$ (expunging bb) $\frac{3ax + 2xx}{6a + 6x} \times cyy = \frac{3a + 2x}{6a + 6x} \times$ Base $\times$ Height.

Hence if $x = o$, then the Solid $= \frac{1}{2}$ Base $\times$ Height, and if x be infinite, then the Solid $= \frac{1}{3}$ Base $\times$ Height: Therefore the Hyperboloid is always between $\frac{1}{2}$ and $\frac{1}{3}$ the circumscribing Cylinder, and is nearly $= \frac{5}{12}$ thereof.

Ex. 8.

133. *Let the Hyperbola CM revolve round the Conjugate AP, A being the Center,* $AC = b$, Semiconjugate $= a$. By the Nature of the Figure $yy = bb + \frac{bb}{aa}xx$; whence $\dot{s} = cyy\dot{x} = cbb\dot{x} + \frac{cbbx^2\dot{x}}{aa}$, and $s = cbbx + \frac{cbb}{3aa}x^3 = \frac{2}{3}cbbx + \frac{1}{3}cyyx$, the Solid described by *ACMP*.

Ex. 9.

Ex. 9.

Let the Hyperbola DM revolve round the Assymptote FIG.
AP, let $AB=b$, $BD=d$, $xy=aa$. Then $\dot{s}=cyy\dot{x}$ 134.
$=\frac{ca^4\dot{x}}{xx}$, whence $s=\frac{-ca^4}{x}$, let the Solid begin at
B; and then the Fluent corrected is $s=\frac{ca^4}{b}-\frac{ca^4}{x}$
$=caa\times\overline{d-y}$.

Note if CAB is not a Right Angle, s must be diminished in the Ratio of the Radius to the Sine of that Angle. When $y=0$, the infinitely long Solid $DMPB$ $=caad=cddb=$ Base $\times AB$.

Or thus: Let $AP=x$, $Pm=y$, $aa=xy$, then $u=x=\frac{aa}{y}$, whence $\dot{s}=2cuy\dot{y}=2caa\dot{y}$; therefore $s=2caay$ $=2cyyx$, for the infinitely long Solid $AHMQP$.

Ex. 10.

Let DM the Log. Curve revolve round the Assymptote 128.
AP, let Subtangent $MP=a$, then $y\dot{x}=a\dot{y}$; and $\dot{s}=$
$cyy\dot{x}=cay\dot{y}$; therefore $s=\frac{cayy}{2}=$ Base $\times\frac{1}{2}TP$, for the infinitely long Solid, $MPAD$.

Ex. 11.

Let AM be a Spheroid, A the Center, $AE=a$, AC 135.
$=b$, then $yy=bb-\frac{bb}{aa}xx$; whence $\dot{s}=cyy\dot{x}=cbb\dot{x}$
$-\frac{cbbx^2\dot{x}}{aa}$; therefore $s=cbbx-\frac{cbb}{3aa}x^3=\frac{2}{3}cbbx$
$+\frac{1}{3}cyyx$. And when $x=a$, then $s=\frac{2}{3}cbba=\frac{2}{3}$ Base $\times$ Height.

Ex. 12.

Let the Segment ADMN of a Circle revolve about the 136.
Sine AN; $CD=r$, $CA=b$, AP or $CQ=x$, $PM=y=$
$\sqrt{rr-xx}-b$; then $\dot{s}=cyy\dot{x}=c\dot{x}\times\overline{rr+bb-xx}-$
$2bc\dot{x}\sqrt{rr-xx}$; and the Fluent $s=cx\times\overline{rr+bb-\frac{1}{3}xx}$
$-2bc\times$ Area $CDMQ$ the Solid described by $ADMP$.

Ex. 13.

FIG. 137. *In the Ciſſoid AM revolving round* $AR=a$, we have $y=\frac{xx}{\sqrt{ax-xx}}$, whence $\dot{s}=cyy\dot{x}=\frac{cx^3\dot{x}}{a-x}=-cx^2\dot{x}$ $-cax\dot{x}-caa\dot{x}+\frac{ca^3\dot{x}}{a-x}$: Therefore (by Form 4 &c) $s=-\frac{cx^3}{3}-\frac{cax^2}{2}-caax-2.30258ca^3\times$ Log. $\overline{a-x}$: And when duly corrected $s=-\frac{1}{3}cx^3-\frac{1}{2}cax^2$ $-caax+2.30258$ Log. $a:-2.30258$ Log. $\overline{a-x}=$ $2.302585ca^3\times$ Log. $\frac{a}{a-x}-\frac{1}{3}cx^3-\frac{1}{2}cax^2-caax$.

Ex. 14.

138. *Let the Ciſſoid revolve round the Aſſymptote AP*, then $x=\frac{\overline{a-y}|^{\frac{3}{2}}}{\sqrt{y}}=u$: Whence $\dot{s}=2cuy\dot{y}=2cxy\dot{y}=$ $2cy^{\frac{1}{2}}\dot{y}\times\overline{a-y}|^{\frac{3}{2}}$: Therefore (by Forms 10, 11 and 13) $s=ca\times$ Segment $ACD+\frac{2}{3}c\times\overline{ay-yy}|^{\frac{3}{2}}$, for the infinitely long Solid $ADMQ$. And the whole $ABMQ$ $=ca\times$ Semicircle ACB.

Ex. 15.

139. *Let the Conchoid DM revolve about the Aſſymptote AP*; $AD=a$, $AC=b$, then $\overline{b+y}\sqrt{aa-yy}=xy$, and $u=x$; therefore $\dot{s}=2cuy\dot{y}=2cxy\dot{y}=2c\dot{y}\times\overline{b+y}\sqrt{aa-yy}=$ $2bc\dot{y}\sqrt{aa-yy}+2cy\dot{y}\sqrt{aa-yy}$. Deſcribe the Quadrant ADK; then (by Forms 10 and 13 &c) $s=2cb\times$ Area $AKLR-\frac{2}{3}c\times\overline{aa-yy}|^{\frac{3}{2}}$. And, duly corrected, $s=2cb\times$ Area $AKLR+\frac{2}{3}ca^3-\frac{2}{3}c\times\overline{aa-yy}|^{\frac{3}{2}}$, for the infinitely long Solid $ARMQP$.

Ex. 16.

139. *Let the Conchoid DM revolve round the Axis DR*, here the Circle deſcribed by $RM=cxx$, and the Fluxion

Fluxion of *DR* or $a-y$ is $-\dot{y}$; therefore the Fluxion FIG. of the Solid $\dot{s} = -cxx\dot{y} = -c\dot{y} \times \frac{\overline{b+y}^2}{yy} \times \overline{aa-yy} =$ $c\dot{y} \times \overline{b+y}^2 - \frac{aabbc\dot{y}}{yy} - \frac{2aabc}{y}\dot{y} - aac\dot{y}$: Whence $s = \frac{c}{3} \times \overline{b+y}^3 + \frac{cbbaa}{y} - 2aabc \times 2.302585 \times$ Log. $y - aacy$. And by due Correction (when $y=a$) $s = \frac{1}{3}c \times \overline{b+y}^3 - \frac{1}{3}c \times \overline{b+a}^3 + \frac{aabbc}{y} - abbc + a^3c$ $- aacy + 2.30258 \times 2caab \times \text{Log.}\frac{a}{y}$.

Ex. 17.

Let the Cycloid AMC revolve about the Axis RC, 129.
let $AR=a$, $AP=x$, $Rp=u$, $pM=y$, Arch $AB=v$, $PB=z$, then $x=a-y$, and $\dot{x}=-\dot{y}$, and by the Nature of the Figure $u=v+z$, and $\dot{u}=\dot{v}+\dot{z}=$ $\frac{a\dot{x}}{2\sqrt{ay-yy}} + \frac{2y-a}{\sqrt{ay+yy}}\dot{x} = \dot{x}\sqrt{\frac{y}{a-y}} = -\dot{y}\sqrt{\frac{y}{a-y}}$: then $\dot{s} = 2cuy\dot{y}$; and assume $s = cuyy + t$, this put into Fluxions we shall find $\dot{t} = -cyy\dot{u} = \frac{cy^{\frac{5}{2}}\dot{y}}{\sqrt{a-y}}$; whence (by Forms the 10th and 11th) $t = \frac{5ca^3}{16}\varphi -$ $\frac{8yy+10ay+15aa}{24} \times c\sqrt{ay-yy}$ (putting $\varphi =$ $\frac{8 \text{ Sectors } CBR}{aa}$): Therefore $s = cuyy + \frac{5}{2}ca \times$ Sector $CBR - \frac{15caa+10acy+8cyy}{24}\sqrt{ay-yy}$, for the Solid described by $PMCR$. And when $y=a$; then $s = \frac{5}{2}ca \times$ Semicircle ABR, the Solid described by the whole Space $AMCR$ revolving about RC.

Ex. 18.

Ex. 18.

FIG. 130. *Let the Catenary AM revolve about the Axis AP;* here $AP=x$, $PM=y$, $AM=z$, and $zz=2ax+xx$, whence (see Ex. 8. Prob. III.) $z\dot{y}=a\dot{x}$. Now $\dot{s}=cyy\dot{x}$, and assume $s=cyyx+t$, this in Fluxions gives $\dot{t}=-2cxy\dot{y}=$ (because $\dot{y}=\frac{a\dot{x}}{z}$) $-2cay\times\frac{x\dot{x}}{z}=$ (because $x\dot{x}=z\dot{z}-a\dot{x}$) $-2cay\times\overline{\dot{z}-\frac{a\dot{x}}{z}}=-2cay\times\overline{\dot{z}-\dot{y}}=2cay\dot{y}-2cay\dot{z}$; then assume $t=cayy-2cayz+u$, this Equation in Fluxions will produce $\dot{u}=2caz\dot{y}=2caa\dot{x}$, whence $u=2caax$. Therefore the Solid $s=cxyy+cayy-2cayz+2caax$.

PROB. XV.

140. *The Nature of the reflecting Curve AMn, and the luminous Point L being given; To find the Focus F, or the Concourse of the nearest reflected Rays MF, nF.*

Take the Particle of the Curve Mn infinitely small, and let C be the Center and CM the Radius of Curvature of the Arch Mn; and on ML, nL, MF, nF let fall the Perpendiculars CE, Ce, CG, Cg; also on the Centers L, F, describe the small Arches Mr, no; then the little Triangles Mon, Mnr are equal and similar, and $Mo=nr$. By the Nature of Reflexion the Angle $LMC=CMF$, and $LnC=CnF$, whence $CE=CG$, and $Ce=Cg$. Now if $CE=Ce$ (that is, if L falls in E or e) then will $CG=Cg$; that is, the Point F will fall in G or g when M and n coincide: But if Ce be less than CE, that is, if L falls below E, then will Cg be less than CG, and the Intersection F will then

then fall above G towards towards M; and the contrary. The Triangles $LE\mathcal{Q}$, LMr; and Fon, FSG are similar, and $E\mathcal{Q} = CE - Ce = CG - Cg = SG$; and $Mr = no$, and $FG = MG - MF = ME - MF$; therefore $LM : LE :: (Mr : E\mathcal{Q} :: no : SG :: FM : FG ::)\ FM : ME - MF$. Whence $MF = \frac{LM \times ME}{2LM - ME}$. FIG. 140.

Otherwise thus: Draw the Tangent MP, and the Perpendicular LP, and let $LM = y$, $LP = u$, $ME = v$, and by Prob. V. $MC = \frac{y\dot{y}}{\dot{u}}$, whence by similar Triangles $vy = \frac{yu\dot{y}}{\dot{u}}$, or $v\dot{u} = u\dot{y}$, therefore $MF = \frac{uy\dot{y}}{2y\dot{u} - u\dot{y}}$.

1. *Wherefore if CM be the Radius of Curvature, CE perpendicular to LM, and LP perpendicular to the Tangent PM, and we make the Distance of the radiating Point* $LM = y$, $ME = v$, $LP = u$: *Then compute the Value of* v *by Prob. V. and take* $MF = \frac{vy}{2y - v}$: *And when AM is convex towards L write* $-v$ *instead of* $+v$.

2. *Or find* $\dot{u}$ *from the Nature of the Curve, by Help of which expunge* $\dot{u}$ *out of the Equation* $MF = \frac{uy\dot{y}}{2y\dot{u} - u\dot{y}}$.

Cor. The Curve FfH passing through all the Points F, or touching all the reflected Rays MF, mf is called the Catacaustic or Caustic by Reflexion. In which any Portion HfF of the Curve is $= LM + MF - LA - AH$. For drawing mL infinitely near ML, and Mo, mr perpendicular to mF, ML; then since $mo = Mr$, therefore $Lm + mf = LM + Mf$, or $LM + Mf - Lm - mf = 0$; and adding fF, $LM + MF - Lm - mf = fF$; but these Moments are as the Fluxions, and therefore the Fluents thereof will be equal, that is the Curve $HF = LM + MF - LA - AH$. 141.

Ex. 1.

Ex. 1.

FIG. 142. *Let AM be a right Line*, then v is infinite, whence $MF = \frac{vy}{2y-v} = \frac{vy}{-v} = -y$.

Or thus: u is a ſtanding Quantity and $\dot{u}=0$, therefore $MF = \frac{u y\dot{y}}{2y\dot{u} - u\dot{y}} = \frac{u y\dot{y}}{-u\dot{y}} = -y$: Whence Perpendicular $PF=PL$, and F is the Focus of the reflected Rays.

Ex. 2.

143. *Let MD be a Circle, C* the Center; then $MF = \frac{vy}{2y-v}$. And when y is infinite, $MF=\frac{1}{2}v=\frac{1}{2}ME$.

And if MD be very ſmall, $MF = \frac{LD \times DC}{2LC + CD}$. And if $LC = CD$, then $MF = \frac{2}{3}v = \frac{2}{3}ME = \frac{1}{3}LM$.

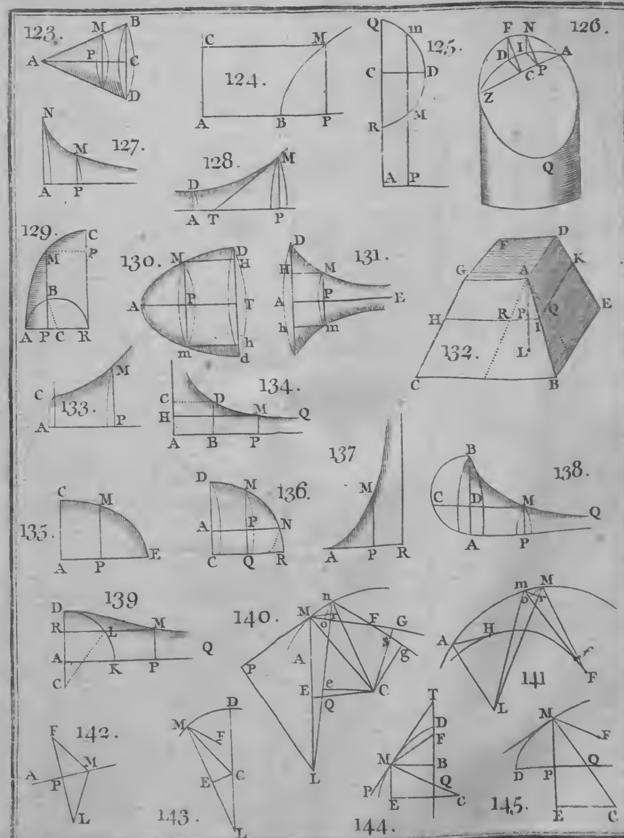
Ex. 3.

144. *Let MD be a Parabola*, and let the Rays be parallel to the Axis DB, then y is infinite, whence $MF = \frac{vy}{2y-v} = \frac{1}{2}v$. But by Prob. V. $v = \frac{2.MQ^2}{r}$ (putting r = Latus Rectum) alſo $QT = \frac{2QM^2}{r}$, therefore $QT=v$, and $QF=\frac{1}{2}v=MF$, and therefore F is the Focus of the Parabola.

Ex. 4.

145. *Let DM be a Parabola*, and let all the Rays be perpendicular to the Axis DQ. Let $DP=x$, $PM=z$, $MQ=s$, and $rx=zz$. Then by Prob. V. ME or $v = \frac{4ssz}{rr}$, and $ss=zz+\frac{1}{4}rr$ by the Nature of the Figure: Whence $MF=\frac{vy}{2y-v} = \frac{1}{2}v = \frac{2z^3}{rr} + \frac{1}{2}z = \frac{2zx}{r} + \frac{1}{2}z$.

Ex. 5.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Ex. 5.

Let DM be the Log. Spiral, the Center L the luminous Point. By Prob. V. $v=y$, whence $MF=\frac{vy}{2y-v}=y$. FIG. 146.

Ex. 6.

Suppose DM to be an Ellipsis, L the Focus, the Transverse $=2r$, Conjugate $=2c$; then by the Property of the Curve PL or $u=\frac{cy}{\sqrt{2ry-yy}}$, and $\dot{u}=\frac{cr\dot{y}}{\overline{2ry-yy}|^{\frac{3}{2}}}$; whence $MF=\frac{uy\dot{y}}{2y\dot{u}-u\dot{y}}=\frac{cy^2\dot{y}\times\overline{2ry-yy}}{cy^3\dot{y}}=2r-y$; consequently (since the Angle $LMP=FMT$) the Point F is the other Focus. 147.

And in like Manner it will be found that Rays issuing from one Focus of an Hyperbola, and reflected by the Curve will diverge from the other Focus.

PROB. XVI.

The Nature of the refracting Curve AMn, and the luminous Point L being given; to find the Focus F, or the Point where the nearest refracted Rays MF, nF concur; and where they meet the Axis of the Figure. 148.

Suppose the Arch Mn to be infinitely small, and let C be the Center and CM the Radius of Curvature in M; and let fall the Perpendiculars CE, Ce on the Rays of Incidence LM, Ln, and the Perpendiculars CG, Cg on the refracted Rays MF, nF; and let the Sine of Incidence CE to the Sine of Refraction CG be as m to n. On the Centers L, F, describe the small Arches Mr Mo. Now since Ce exceeds CE, therefore

FIG. Cg exceeds CG, they being in a given Ratio; whence MF, nF interfect beyond G.

The Figures $GCME$ and $onMr$ are fimilar; therefore $ME : MG :: Mr : Mo = \frac{MG \times Mr}{ME}$; and $LM : LQ :: Mr : Qe = \frac{LQ \times Mr}{LM}$: And by the Property of Refraction, $m : n :: (Ce : Cg :: CE : CG :: Ce - CE : Cg - CG ::)\ eQ : Sg = \frac{n \times LQ \times Mr}{m \times LM}$; and by the fimilar Triangles FoM, Fgs, $Mo - Sg : Mo :: MG : MF = \frac{m \times MG^2 \times LM}{m \times MG \times LM - n \times LQ \times ME}$.

149. Produce MF till it interfect the Axis of the Curve in O, and let $LA = d$, $AO = f$, $LM = y$, $MO = s$, $AH = x$, $MH = z$, then $y = \sqrt{z^2 + \overline{d+x}^2}$, and $s =$

148. $\sqrt{zz + \overline{f-x}^2}$. And it is $\dot{y} : -\dot{s} :: rn : on :: m : n$, and $n\dot{y} = -m\dot{s}$ or $n\dot{y} = -m\dot{s}$, that is $\frac{nz\dot{z} + nd\dot{x} + nx\dot{x}}{\sqrt{zz + \overline{d+x}^2}} = \frac{mf\dot{x} - mx\dot{x} - mz\dot{z}}{\sqrt{zz + \overline{f-x}^2}}$. Therefore,

149. 1. *To find the Focus F, let CM be the Radius of Curvature, CE perpendicular to LM, and CG to MG,* $ME = v$, $MG = u$. $LM = y$, *m and n the Sines of Incidence and Refraction. Then find v and u by Prob. V. and take* $MF = \frac{muuy}{muy - nvv - nvy}$. *If AM is concave towards L, write* $-v$ *for* v, *and* $-u$ *for* u: *And if the Rays converge when they fall on the Curve AM, write* $-y$ *for* y.

2. *To find the Point O where the refracted Ray meets the Axis of the Curve, let* $LA = d$, $AO = f$, $AH = x$, $MH = z$, *then by the Nature of the Curve expunge* $\dot{x}$ *or* $\dot{z}$ *out of the Equation* $\frac{nz\dot{z} + nd\dot{x} + nx\dot{x}}{\sqrt{zz + \overline{d+x}^2}} = \frac{mf\dot{x} - mx\dot{x} - mz\dot{z}}{\sqrt{zz + \overline{f-x}^2}}$; *and by Reduction find f. And when*

when the Curve is concave towards L, write $-x$ for $+x$, &c. FIG.

COR. The Curve NfF passing through all the Points F, or which touches all the refracted Rays KN, MF, is called the Diacaustic or Caustic by Refraction. And any Portion of it $NF = FM + \frac{n}{m}ML - NK - \frac{n}{m}KL$. For supposing Mn infinitely small, and drawing Mo, Mr, Perpendiculars on fn, nL; then by the Nature of Refraction $m : n :: rn : on = \frac{n}{m}rn$, therefore $on = \frac{n}{m}rn = 0$; that is $Mf - nf - \frac{n}{m} \times \overline{Ln - LM} = 0$, and adding Ff; we have $Ff = MF - nf - \frac{n}{m} \times \overline{Ln - LM}$; but these Moments are as the Fluxions, whence the Fluents will be equal, or $FN = FM - KN - \frac{n}{m} \times \overline{LK - LM}$. 150.

Example 1.

Let AM be a plane Surface, then v, u are infinite, 151.
whence $\frac{muuy}{muy - nvv - nvy} = \frac{muuy}{-nvv}$. Now let $r =$ the infinite Radius of Curvature, the Perpendicular $LA = p$, then by the similar Triangles LAM, MCE, $v = \frac{rp}{y}$; also $mm \times \overline{rr - uu} = nn \times \overline{rr - vv}$, by the right angled Triangles MEC, MGC: Whence $uu = \frac{mm - nn}{mm} rr + \frac{nn}{m^2} vv$; therefore $MF = \frac{-muuy}{nvv} = -\frac{mm - nn}{mnvv} \times rry - \frac{ny}{m} = -\frac{m^2 - n^2}{mnp^2} y^3 - \frac{n}{m} y$. And near the Point A, $Af = -\frac{m}{n} p$.

Ex. 2.

FIG. 152. *Let parallel Rays fall on the convex Side of the Sphere AM*; then y is infinite, and $MF = \frac{muuy}{muy - nvv - nvy} = \frac{muu}{mu - nv}$, and near the Vertex A, $u = v = AC$; whence $AF = \frac{m \times AC}{m-n}$.

Ex. 3.

153. *Let parallel Rays fall on the concave Side of the Sphere AM*; then y is infinite, and v, u are negative; whence $MF = \frac{muuy}{-muy - nvv + nvy} = \frac{muu}{nv - mu}$, and at the Vertex A, $v = u = AC$, then $AF = \frac{mu}{n-m} = \frac{m}{n-m} AC$.

Ex. 4.

154. *Suppose the Rays proceeding from L, to fall on the convex Side of the Sphere AM*; then $MF = \frac{muuy}{muy - nvv - nvy}$. And in or near the Vertex, $MF = \frac{muy}{\overline{m-n} \times y - nu} = \frac{mry}{\overline{m-n} \times y - nr}$.

And to find where the refracted Ray meets the Axis of the Sphere, $zz = 2rx - xx$, and $z\dot{z} = r\dot{x} - x\dot{x}$, by which expunge z and $\dot{z}$ out of the Equation $\frac{nz\dot{z} + nd\dot{x} + nx\dot{x}}{\sqrt{zz + \overline{d+x}^2}} = \frac{mf\dot{x} - mx\dot{x} - mz\dot{z}}{\sqrt{zz + \overline{f-x}^2}}$, and there arises $\frac{nr + nd}{\sqrt{2rx + 2dx + dd}} = \frac{mf - mr}{\sqrt{2rx - 2fx + ff}}$, by Reduction of which f is found. And in the Vertex A where x is 0, $\frac{mf - mr}{f} = \frac{nr + nd}{d}$, or $f = \frac{mdr}{\overline{m-n} \times d - nr}$.

Ex. 5.

Ex. 5.

If Rays fall on the concave Surface of the Sphere AM, then $MF = \frac{muuy}{-muy-nvv+nvy}$. And near the Vertex, $MF = \frac{muy}{\overline{n-m} \times y-nu} = \frac{mry}{\overline{n-m} \times y-nr}$. FIG. 155.

To find where MF cuts the Axis, here $zz=2rx - xx$; put the Equation into Fluxions, and write $-x$ for x, $-\dot{x}$ for $\dot{x}$, $-r$ for r in the Equation $\frac{nz\dot{z}+nd\dot{x}+nx\dot{x}}{\sqrt{zz+\overline{d+x}^2}} = \frac{mf\dot{x}-mx\dot{x}-mz\dot{z}}{\sqrt{zz+\overline{f-x}^2}}$, and we have $\frac{nr-nd}{\sqrt{dd-2dx+2rx}} = \frac{-mr-mf}{\sqrt{ff+2fx+2rx}}$; and in the Vertex $\frac{nr-nd}{d} = \frac{-mr-mf}{f}$, or $f = \frac{mrd}{\overline{n-m} \times d-nr}$.

Cor. Since by the foregoing Equation $n^2 \times \overline{r-d}^2 \times \overline{ff+2fx+2rx} = m^2 \times \overline{r+f}^2 \times \overline{dd-2dx+2rx}$, and in the Vertex $n^2 \times \overline{r-d}^2 = m^2 \times \overline{f+r}^2$, if $d = \frac{m+n}{n} r$; then, in the Vertex, $f = -\frac{m+n}{m} r$, and since the same Value of d and f substituted in the other Part of the Equation $n^2 \times \overline{r-d}^2 \times \overline{2fx+2rx} = m^2 \times \overline{r+f}^2 \times \overline{2rx-2dx}$, or $n^2 \times \overline{r-d} = m^2 \times \overline{r+f}$, will still make both Sides of the Equation equal: Therefore it follows, that when $d = \frac{m+n}{n} r$, then will $f = -\frac{m+n}{m} r$, for all Rays fall on all Parts of the Sphere; and therefore F or O will be a geometrical Focus.

And on the contrary, Rays converging to O, and falling on the Sphere, will all be accurately refracted to their Focus L.

Ex. 6.

Ex. 6.

FIG. 156. *Let parallel Rays fall upon the Spheroid AM,* let Tranſverſe $= b$, Parameter $= a$, then $zz = ax - \frac{a}{b}xx$, and $z\dot{z} = \frac{1}{2}a\dot{x} - \frac{a}{b}x\dot{x}$; and ſince d is infinite, therefore the Equation for AO becomes $n = \frac{mf - mx - \frac{ma}{2} + \frac{ma}{b}x}{\sqrt{ax - \frac{a}{b}xx + \overline{f-x}^2}}$, whence f will be found; and in the Vertex where $x = 0$, $mf - \frac{1}{2}ma = nf$, or $f = \frac{\frac{1}{2}ma}{m-n}$.

COR. If $b : a :: m^2 : m^2 - n^2$, then $a = \frac{b \times \overline{m^2 - n^2}}{m^2}$, and $\frac{nb}{2m} =$ Diſtance of the Focus from the Center, then $f = \frac{m}{m-n} \times \frac{1}{2}a = \frac{m+n}{2m} \times b = \frac{1}{2}b + \frac{bn}{2m} =$ Diſtance of the remoter Focus from the Vertex; and ſince the ſame Values of a and f ſubſtituted into the general Equation $n\sqrt{ax - \frac{a}{b}xx + \overline{f-x}^2} = mf - \frac{ma}{2} - mx + \frac{ma}{b}x$, keep both Sides equal; therefore when parallel Rays fall upon a Spheroid in a Direction parallel to the Axis, and $a = \frac{b \times \overline{m^2 - n^2}}{m^2}$, then all theſe Rays falling on all Points of the Spheroid will be accurately refracted to the further Focus of the Figure.

And on the contrary, Rays iſſuing from the further Focus of this Spheroid and refracted at the Surface, will all emerge parallel to the Axis.

PROB.

PROB. XVII.

FIG.

To find the Center of Gravity of any Line, Surface or Solid.

The Center of Gravity of a Body is that Point upon which, if it were suspended, it would rest in any given Position.

Let MN be any Figure or Solid Body, regular or irregular; C it's Center of Gravity; and suppose it to be suspended in C upon the horizontal Line SC, and the Axis of Suspension to pass through S, parallel to the Horizon and perpendicular to SC. Let all the infinitely small Particles of the Body be reduced to the Line SC, situated respectively in Planes perpendicular to SC; and let the Magnitude of every two Particles of the Body taken on different Sides of C (as a and g, b and h, d and i, &c) be reciprocally as their Distances from C; then is C still the Center of Gravity of each corresponding two, as it is of the whole MN. Therefore we have $Ca \times a = Cg \times g$, that is $\overline{SC - Sa} \times a = \overline{Sg - SC} \times g$, or $\overline{a+g} \times SC = Sa \times a + Sg \times g$. In like Manner $\overline{b+h} \times SC = Sb \times b + Sh \times h$, and $\overline{d+i} \times SC = Sd \times d + Si \times i$ &c, whence $\overline{a+b+d+e+g+h+i+k} \times SC = Sa \times a + Sb \times b, + Sd \times d + Se \times e + Sg \times g + Sh \times h + Si \times i + Sk \times k$, &c. 157.

consequently $SC = \dfrac{Sa \times a + Sg \times g + Sb \times b + Sh \times h \text{ &c}}{a + g + b + h \text{ &c}}$.

Now if any one of the variable Distances, as Sb be called x, the Body MN, s; then will $Sb \times b = x\dot{s}$, and the Sum of all the $Sa \times a + Sb \times b$ &c $=$ Sum of all the $x\dot{s}$, or the Fluent of $x\dot{s}$: And the Sum of all the $a + b + g$ &c $=$ Sum of all the $\dot{s}$, or the Fluent

FIG. Fluent of $\dot{s}$ that is the Body MN; therefore $SC = \frac{\text{Fluent of } x\dot{s}}{\text{Body } MN \text{ or } s}$. Therefore

To find the Center of Gravity; let $s =$ Line, Surface or Solid: Multiply the Fluxion of the Line, Surface or Solid ($\dot{s}$) by the Distance (of the Center of Gravity of the generating Point, Line or Plane) from the Axis of Suspension; and find the Fluent z; then $\frac{z}{s} =$ Distance of the Center of Gravity from the Point of Suspension.

Ex. 1.

158. *Let SB be a right Line or Cylinder, S the Point of* Suspension; $SB = x$; then $\dot{z} = x\dot{x}$, and $z = \frac{1}{2}xx$; therefore $\frac{z}{s} = \frac{1}{2}x$, for the Distance of the Center of Gravity SC.

Ex. 2.

159. *In the Triangle SQD*, whose Point of Suspension is S; let SF bissect the opposite Side QD, then the Center C is in the Line SF; draw AE parallel, and SG perpendicular to QD, put $SF = a$, $SB = x$, $SI = v$, $SG = h$, $QD = b$, $AE = y$. By similar Triangles $v = \frac{hx}{a}$, and $\dot{v} = \frac{h\dot{x}}{a}$, $y = \frac{bx}{a}$; then $\dot{z} = xy\dot{v} = \frac{bhx^2\dot{x}}{aa}$; and $z = \frac{bhx^3}{3aa}$. Also $\dot{s} = y\dot{v} = \frac{bhx\dot{x}}{aa}$, and $s = \frac{bhx^2}{2aa}$: Therefore $\frac{z}{s} = \frac{2}{3}x$, and when $x = a$, $\frac{z}{s} = \frac{2}{3}a = SC$.

Ex. 3.

160. *Let AM be the Arch of a Circle, s its* Center, $AD = b$, $SE = r$, $SB = x$, $AM = v$, $BM = y$. It is evident the Center of Gravity of any Arch AEG is in the Line SE that bissects it. Whence $\dot{z} = x\dot{v} =$ (by the Nature of the Circle) $-r\dot{y}$; and $z = -ry$; and the Fluent

Fluent corrected is $z = bs - ry$. Whence $\frac{z}{s} = \frac{rb - ry}{v}$, and when $y = 0$, $\frac{z}{s} = \frac{rb}{v} = SC$, the Distance of the Center of Gravity of the Arch AEG from S. FIG.

Ex. 4.

For the Sector of a Circle MmS, whose Center and Point of Suspension is S; let Arch $Mm = c$, Radius $SM = r$, $Mm = a$, $SQ = x$, $QDq = v$. Then $Qq = \frac{ax}{r}$, and by the last Example, the Distance of the Center of Gravity of the Arch QDq from S is $= \frac{ax^2}{rv}$, therefore $\dot{z} = \frac{ax^2\dot{x}}{r}$, and $z = \frac{ax^3}{3r}$; therefore $\frac{z}{s} = \frac{ax^3}{3r \times \frac{1}{2}xv} = \frac{2ax^2}{3rv} = \frac{2ax}{3c}$; and when $x = r$, then $SC = \frac{2ar}{3c}$. 161.

Ex. 5.

For the Circular Area PQD. Let SD or $SE = r$, $SQ = b$, $PQ = c$, $SB = x$, $PA = v$, $AB = y$, by the Nature of the Circle $y = \sqrt{rr - xx}$; then $\dot{z} = yx\dot{x} = x\dot{x}\sqrt{rr - xx}$, and $z = -\frac{1}{3} \times \overline{rr - xx}|^{\frac{3}{2}}$; corrected $z = \frac{\overline{rr - bb}|^{\frac{3}{2}} - \overline{rr - xx}|^{\frac{3}{2}}}{3} = \frac{c^3 - y^3}{3}$. Also $s =$ Area $PABQ = \frac{vr + xy - cb}{2}$; whence $\frac{z}{s} = \frac{2}{3} \times \frac{c^3 - y^3}{vr + xy - cb} = Sc$. And the Distance of the Center of Gravity of the whole $PADQ$ from S is $= \frac{2c^3}{3vr - 3cb}$. 162.

Again in Respect of the Axis of Suspension SD; since the Center of Gravity of the describing Line y, is in the Middle of BA, therefore $\dot{z} = \frac{1}{2}y \times y\dot{x} = \frac{rr\dot{x} - x^2\dot{x}}{2}$, whence $z = \frac{r^2x}{2} - \frac{x^3}{3}$. But (in Q, $x = b$,

FIG. $x=b$, $z=0$) the Fluent corrected is $z=\frac{3rrx-3rrb-x^3+b^3}{6}$; therefore $\frac{z}{s}=\frac{1}{3}\times\frac{3r^2x-3r^2b-x^3+b^3}{vr+xy-cb}$; and when $x=r$, $\frac{2r^3-3r^2b+b^3}{3rv-3cb}=$ Distance of the Center of Gravity of the Semi-segment $PADQ$ from QD.

Ex. 6.

163. *In the Parabola* $rx=yy$, let $SP=x$, $PM=y$, then in Respect of the Axis of Suspension ST, $\dot{z}=yx\dot{x}=x\dot{x}\sqrt{rx}$; and $z=\frac{2}{5}xx\sqrt{rx}$: And $s=\frac{2}{3}xy=\frac{2}{3}x\sqrt{rx}$; therefore $\frac{z}{s}=\frac{3}{5}x$ the Distance of the Center of Gravity from ST.

Again in Regard to the Axis of Suspension SP, because the Center of Gravity of the describing Line is in the Middle of MP, therefore $\dot{z}=\frac{1}{2}yy\dot{x}=\frac{y^3\dot{y}}{r}$, and thence $z=\frac{y^4}{4r}$; and $s=\frac{2y^3}{3r}$: Therefore $\frac{z}{s}=\frac{3}{8}y$, the Distance from SP.

Ex. 7.

164. *For the hyperbolic Area BCMP, between the Asymptotes.* Let $SP=b$, $BC=c$, $SP=x$, $PM=y$; $cb=xy$. Then in Regard to the Axis SD, $\dot{z}=yx\dot{x}=cb\dot{x}$, and $z=cbx$, but in B, $x=b$, therefore by Correction $z=cb\times\overline{x-b}$. And $\frac{z}{s}=\frac{cb\times\overline{x-b}}{\text{Area } BCMP}$.

Again for the Axis of Suspension SP, $\dot{z}=\frac{1}{2}yy\dot{x}=\frac{ccbb}{2xx}\dot{x}$, and $z=\frac{-ccbb}{2x}=\frac{-cby}{2}$; corrected $z=\frac{cb}{2}\times\overline{c-y}$ and $\frac{z}{s}=\frac{cb\times\overline{c-y}}{2\text{ Area } BCMP}$.

Ex. 8.

Ex. 8.

Let AMB be an Ellipsis, S the Center, $AS=a$, $SB=b$, $SQ=y$, $QM=x=\frac{a}{b}\sqrt{bb-yy}$, then for the elliptic Space, $\dot{z}=x\dot{y}=\frac{ay\dot{y}}{b}\sqrt{bb-yy}$; and $z = -\frac{a}{3b}\times\overline{bb-yy}|^{\frac{3}{2}}$, corrected $z=\frac{abb}{3}-\frac{a}{3b}\times\overline{bb-yy}|^{\frac{3}{2}}$: Thence $\frac{z}{s}=\frac{ab^3-a\times\overline{bb-yy}|^{\frac{3}{2}}}{3b\times\text{Area } SAMQ}$. FIG. 165.

Likewise for the Distance from SB, $\dot{z}=\frac{1}{2}xx\dot{y}=\frac{aa\dot{y}}{2}-\frac{aay^2\dot{y}}{2bb}$, and $z=\frac{aay}{2}-\frac{aay^3}{6bb}$; whence $\frac{z}{s}=\frac{3bbay-aay^3}{6bb\times\text{Area } SAMQ}$.

Ex. 9.

Let SMP be the hyperbolic Space, Transverse $=2a$, Conjugate $=2b$, $SP=x$, $PM=y=\frac{b}{a}\sqrt{2ax+xx}$, whence $xx=2aa+\frac{aa}{bb}yy-\frac{2aa}{b}\sqrt{bb+yy}$, $x\dot{x}=\frac{aa}{bb}y\dot{y}-\frac{aay\dot{y}}{b\sqrt{bb+yy}}$; whence $\dot{z}=yx\dot{x}=\frac{aa}{bb}y^2\dot{y}-\frac{aay^2\dot{y}}{b\sqrt{bb+yy}}$, and $z=\frac{aay^3}{3bb}-\frac{aay}{2b}\sqrt{bb+yy}+\frac{aab}{2}\times 2.30258\,\text{Log.}\,y+\sqrt{bb+yy}:$, whence $\frac{z}{s}=\frac{z}{\text{Area } SMP}=$ Distance from ST. 163.

Then for the Distance from SP, we have $\dot{z}=\frac{yy\dot{x}}{2}=\frac{bb\dot{x}}{2aa}\times\overline{2ax+xx}$, and the Fluent $z=\frac{bbxx}{2a}+\frac{bbx^3}{6aa}$; therefore $\frac{z}{s}=\frac{3abbxx+bbx^3}{6aa\times\text{Area } SPM}$.

Ex. 10.

Ex. 10.

FIG. 166. *For the Surface of a right Cone*, let $SD=f$, $c=$ Circumference of the Bafe, Axis $SB=d$, $SM=v$, $SF=x$. Then it is plain it's Center of Gravity is in the Axis SB. $\frac{cx}{d}=$ Circumference of the Circle MQ; and by fimilar Triangles $v=\frac{fx}{d}$, and $\dot{v}=\frac{f\dot{x}}{d}$; therefore $\dot{z}=\frac{cx^2\dot{v}}{d}=\frac{cfx^2\dot{x}}{dd}$; and $z=\frac{cfx^3}{3dd}$; alfo $s=\frac{cxv}{2d}=\frac{cfxx}{2dd}$: Therefore $\frac{z}{s}=\frac{2}{3}x=SC$.

Ex. 11.

166. *For the Cone (or Pyramid) SDE*, let the Bafe $=b$, the reft as in the laft Example; then the Circle MQ $=\frac{bxx}{dd}$; and $\dot{z}=\frac{bx^3\dot{x}}{dd}$, and $z=\frac{bx^4}{4dd}$; alfo s $=\frac{bx^3}{3dd}$: Therefore $\frac{z}{s}=\frac{3}{4}x=SC$ the Diftance of the Center of Gravity from S.

Ex. 12.

167. *Let SMD be a Sphere*, $SP=x$, $PM=y$, Radius $=r$, $c=3.1416$, then $y=\sqrt{2rx-xx}$, and $\dot{z}=cy^2x\dot{x}$ $=2crxx\dot{x}-cx^3\dot{x}$; therefore $z=\frac{2crx^3}{3}-\frac{cx^4}{4}$: and $s=crx^2-\frac{cx^3}{3}$ (by Ex. 3. Prob. XIV.); therefore $\frac{z}{s}=\frac{8rx-3xx}{12r-4x}$ for the Diftance of the Center of Gravity from S.

Ex. 13.

167. *For the Spheroid SMD, whofe Center is C*, let $SC=a$, $CF=b$, $SP=x$, $PM=y$, $c=3.1416$, then $yy=\frac{bb}{aa}\times\overline{2ax-xx}$; and $\dot{z}=cyyx\dot{x}=\frac{cbb}{aa}\times\overline{2ax\dot{x}-x^2\dot{x}}$; and

and $z = \frac{cbb}{aa} \times \overline{\frac{2}{3}ax^3 - \frac{1}{4}x^4}$; and $s = \frac{cbb}{aa} \times \overline{ax^2 - \frac{1}{3}x^3}$; FIG.

and $\frac{z}{s} = \frac{8ax - 3xx}{12a - 4x}$.

Ex. 14.

To find the Center of Gravity of the Solid SBDm, generated by a partial Revolution of the Parabola SMD about the Axis SB. 168.

Let s be the Point of Suspension, let $SB = d$, $BD = b$, $SP = x$, $PM = y, ax = yy$, Arch $Dd = c$; then Arch $Mm = \frac{cy}{b}$; therefore $\dot{z} = \frac{cyy}{2b}x\dot{x} = \frac{cax^2\dot{x}}{2b}$; and $z = \frac{cax^3}{6b}$: and $s = \frac{cax^2}{4b}$, therefore $\frac{z}{s} = \frac{2}{3}x$, the Distance from ST.

Again for it's Distance from SB, let Chord $Dd = f$, then Chord $Mm = \frac{fy}{b}$; and by Ex. 4. the Distance of the Center of Gravity of the Sector PMm from $P = \frac{2fy}{3c}$; therefore $\dot{z} = \frac{fy^3\dot{x}}{3b} = \frac{2fy^4\dot{y}}{3ab}$; and $z = \frac{2fy^5}{15ab}$; therefore $\frac{z}{s} = \frac{8fy^5}{15caax^2} = \frac{8fy}{15c}$, the Distance of the Center of Gravity from SP, and in the Plane that passes through the Axis and bissects the Base.

Ex. 15.

Let the Hyperbola CM revolve round the Assymptote SP, and describe an Hyperboloid CMB: Let $SB = b$, $BC = d$, $SP = x$, $PM = y$, $c = 3,1416$. $bd = xy$; and $\dot{z} = cyyx\dot{x} = \frac{bbddc\dot{x}}{x}$, whence $z = bbddc$ Log. x: 169.

And corrected $z = bbddc$ Log. $\frac{x}{b}$: Also $\dot{s} = cyy\dot{x} = \frac{cbbdd\dot{x}}{xx}$; and $s = -\frac{cbbdd}{x}$, and corrected $s =$

cbbdd

FIG. $cbbdd \times \frac{x-b}{bx}$; therefore $\frac{z}{s} = \frac{bx}{x-b} \times \text{Log.} \frac{x}{b}$ = Diſtance of the Center of Gravity of the Solid from SA.

Ex. 16.

163. *Let the Solid be an Hyperboloid*, Tranſverſe $= 2a$, Conjugate $= 2b$, $c = 3.1416$, $SP = x$, $PM = y$, $= \frac{b}{a}\sqrt{2ax+xx}$; whence $\dot{z} = cyyx\dot{x} = \frac{cbb\dot{x}}{aa} \times \overline{2ax^2+x^3}$; whence $z = \frac{cbb}{aa} \times \overline{\tfrac{2}{3}ax^3+\tfrac{1}{4}x^4}$; alſo $\dot{s} = \frac{cbb}{aa}\dot{x} \times \overline{2ax+xx}$; and $s = \frac{cbb}{aa} \times \overline{axx+\tfrac{1}{3}x^3}$; therefore $\frac{z}{s} = \frac{8ax+3xx}{12a+4x}$ = Diſtance of the Center of Gravity from the Vertex S.

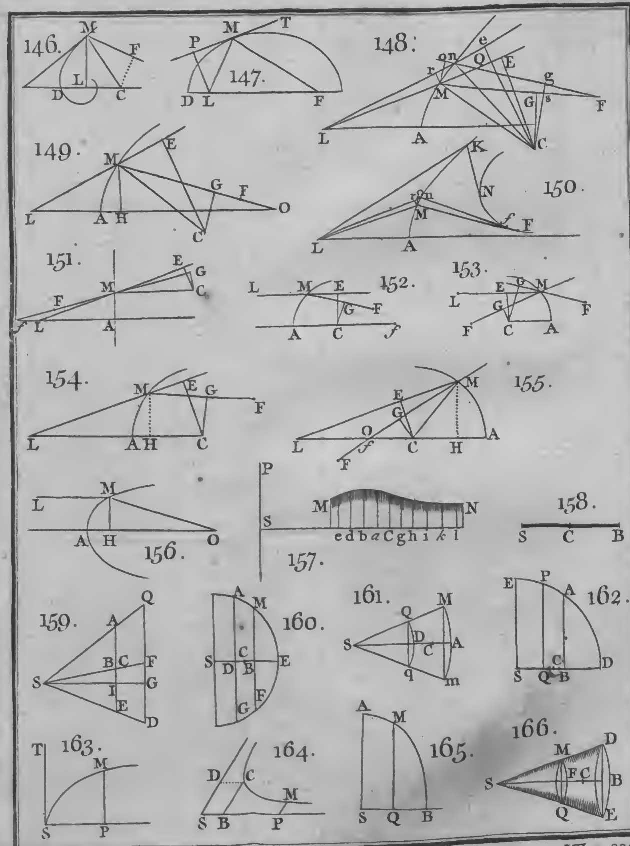
PROB. XVIII.

To find the Centers of Percuſſion and Oſcillation.

The Center of Percuſſion is that Point in the Axis of a vibrating Body, which ſtriking againſt an immovable Obſtacle, the Body ſhall incline to neither Side, but reſt as it were in Equilibrio, on that Point.

And the Center of Oſcillation is the Point in the Axis of a vibrating Body, in which if a ſmall Body or Particle be placed, it ſhall perform it's Vibrations after the ſame Manner, in the ſame Time, and with the ſame angular Velocity as the whole Body.

170. To find the Center of Percuſſion; through the Point of Suſpenſion C, and Center of Gravity, draw the Axis of the Body CO; and ſuppoſe O to be the Center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Center of Percussion; through CO draw the Plane in FIG. which the Center of Gravity moves, and imagine the Body to be divided into innumerable small Prisms, all perpendicular to this Plane, and let them be supposed to be reduced to, or situated in, the Points where they intersect this Plane; and let p be one of these small Prisms. Draw pf perpendicular to CO, and pd perpendicular to Cp; then pd will be the Direction of p's Motion as it revolves about C; and the Body being stopt at O, p will urge the Point d forward, with a Force proportional to it's Magnitude and Velocity, that is as $p \times Cp$; therefore the Force wherewith p acts at d in a Direction perpendicular to CO, will be $p \times Cf$. And the Force by which p endeavours to turn the Body about O, will be as $p \times Cf \times do$, or $p \times Cf \times \overline{CO - Cd}$, that is as $p \times Cf \times CO - p \times Cp^2$. Now since the Sum of all these Forces to turn the Body about O must be $= 0$, therefore all the $p \times Cf \times CO - p \times Cp^2 = 0$, or all the $p \times Cf \times CO =$ all the $p \times Cp^2$; therefore $CO = \frac{\text{Sum of all the } p \times Cp^2}{\text{Sum of all the } p \times cf}$.

For the Center of Oscillation. Through the Centers of Motion C and Gravity G draw the Axis CO, and let O be the Center of Oscillation. Draw the horizontal Line Cr, and Or, Gg, pn perpendicular thereto, and pf perpendicular to CO. 171.

By Reason of the equal angular Velocities of all the Particles of the Body; the absolute Motion of any Particle q (and consequently the Force that generates it) will be as $Cq \times q$; and a Force acting at n that can generate that Motion in q is as $\frac{Cq^2 \times q}{Cn}$; let $s =$ Sum of all the $Cq^2 \times q$ in the Body, and let this be as (the Weight of) the Particle p. If the Weight of the Particle p generates any Motion in the whole Body AD, then $\frac{Cq^2 \times q}{s} p =$ that Part of the Gravity of p which generates the Motion of the Particle q; and the

FIG. 171. the Motion of q generated by that Force is $\frac{Cn}{Cq} \times \frac{Cq^2 \times q}{s} p$ or $\frac{Cn \times Cq \times q}{s} p$ and the Velocity of q is $\frac{Cn \times Cq \times p}{s}$, and its angular Velocity $\frac{Cn \times p}{s}$. After the ſame Manner any other Force $\frac{Cq^2 \times q}{s} p$ acting at n, will generate the ſame angular Velocity in any other Particle q; and conſequently the Sum of all the Forces $\frac{Cq^2 \times q}{s} p$ or the Weight p will generate the ſame angular Velocity in all the Particles q together or in the whole Body. Now ſince the Weight of any Particle p will generate an angular Velocity in the Body AD about C, which is as $\frac{Cn \times p}{\text{Sum of all the } Cp^2 \times p}$; therefore the angular Velocity which all the Particles p can generate is as $\frac{\text{the Sum of all the } Cn \times p}{\text{Sum of all the } Cp^2 \times p}$. In like Manner the angular Velocity which the Gravity of a Particle p placed in O would generate in itſelf is as $\frac{Cr \times p}{CO^2 \times p}$ or as $\frac{Cr}{CO^2}$. But becauſe of the Equality of the Vibrations and correſpondent Accelerations, this laſt muſt be equal to the Sum of all the former; whence $\frac{\text{Sum of } Cn \times p}{\text{Sum of } Cp^2 \times p} = \frac{Cr}{CO^2}$: But by the Nature of the Center of Gravity, the Sum of all the $Cn \times p = Cg \times \text{Body } AD = \frac{Cg}{CG} \times CG \times \text{Body } AD = \frac{Cr}{CO} \times \text{Sum of all the } Cf \times p$. Therefore $\frac{Cr}{CO} \times \frac{\text{Sum } Cf \times p}{\text{Sum } Cp^2 \times p} = \frac{Cr}{CO^2}$; whence $CO = \frac{\text{Sum of all } Cp^2 \times p}{\text{Sum of all } Cf \times p}$: And therefore the Center of Oſcillation is the ſame with the Center of Percuſſion.

Since

FIG. 171.

Since Sum of $Cf \times p = CG \times$ Body AD; therefore $CO = \frac{\text{Sum of } Cp^2 \times p}{CG \times \text{Body } AD}$.

Also since $Cp^2 = CG^2 + Gp^2 - 2CG \times Gf$, therefore the Sum of all $Cp^2 \times p =$ Sum of all $\overline{GC^2 + Gp^2} \times p$ - Sum of all $2CG \times Gf \times p$. But by the Nature of the Center of Gravity, the Sum of all the $Gf \times p = 0$; therefore Sum $Cp^2 \times p =$ Sum $CG^2 \times p +$ Sum $Gp^2 \times p = CG^2 \times$ Body $AD +$ Sum $Gp^2 \times p$. Therefore $CO = \frac{\text{Sum } Cp^2 \times p}{CG \times \text{Body} AD} = CG + \frac{\text{Sum of } Gp^2 \times p}{CG \times \text{Body} AD}$. And $GO = \frac{\text{Sum of all } Gp^2 \times p}{CG \times \text{Body} AD}$.

Therefore if $s =$ Body, $Cp = x$, $Cf = v$, $Gp = z$, then the Sum of all the $Cp^2 \times p =$ Sum of $x^2 s' =$ Fluent of $x^2 \dot{s}$, and Sum of $Cf \times p =$ Sum of $vs' =$ Fluent of $v\dot{s}$. And Sum of $Gp^2 \times p =$ Sum $z^2 s' =$ Fluent of $z^2 \dot{s}$; therefore $CO = \frac{\text{Fluent of } x^2 \dot{s}}{\text{Fluent of } v\dot{s}} = \frac{\text{Fluent of } x^2 \dot{s}}{CG \times \text{Body} AD}$ $= CG + \frac{\text{Fluent of } z^2 \dot{s}}{CG \times \text{Body} AD}$.

RULE.

Suppose the Body to be described or generated by a Plane perpendicular to the Axis of the Body, or parallel to the Axis of Motion. Then

1. *Muliply the Fluxion (or if the Case require, the second Fluxion) of the Body by the Square of the Distance (of the generating Plane) from the Axis of Motion, and find the Fluent (once or twice, as there is Occasion) by Help of the Equation of the Figure; which call F.*

Or thus: Imagine a Line drawn through the Center of Gravity of the Figure parallel to the Axis of Motion; then multiply the Fluxion (or second Fluxion) of the Body into the Square of the Distance (of the generating Plane &c) from this Line, and by Help of the Equation of the Figure, find the Fluent (once or twice) which call G.

FIG. 2. *Multiply the Fluxion of the Body by the Distance from the Point of Suspension to the Point where the generating Plane (or Line &c) cuts the Axis of the Body; and find the Fluent M. And let $d =$ Distance from the Point of Suspension to the Center of Gravity, and $B =$ the Body. Then $\frac{F}{M}$ or $\frac{F}{dB}$ or $d + \frac{G}{dB}$ will be the Distance of the Center of Percussion or Oscillation from the Point of Suspension.*

SCHOLIUM.

If the Center of Percussion or Oscillation be made the Center of Suspension, then the former Point or Center of Suspension becomes the Center of Percussion; if the Plane of it's Motion remain the same. For $\frac{G}{B} = d \times$ Distance of the Centers of Gravity and Percussion.

COR. The Center of Pressure of any Plane immersed in a Fluid and sustaining that Fluid, is the same with the Center of Percussion of that Plane; the Axis of Motion being the Intersection of this Plane with the Surface of the Fluid. The Center of Pressure is that Point against which a Force being applied $=$ Sum of all the Pressures, shall just sustain them; so as the Plane shall incline to neither Side.

172. Through the Center of Gravity of the Plane draw AO perpendicular to AS the Intersection of the Plane and Surface of the Fluid, and let cd be parallel to AS. Then the Pressure against any small Part cd is as $cd \times Ab$, and it's Force to turn the Plane about O the Center of Pressure is $cd \times Ab \times bO = cd \times Ab \times AO - cd \times Ab^2$, and the sum of all these must be equal to o, therefore $AO = \frac{\text{Sum of } cd \times Ab^2}{\text{Sum of } cd \times Ab}$, therefore O is the same with the Center of Oscillation and Percussion, and consequently is to be found the same Way.

Example 1.

Example 1.

Let CB be a right Line, $CB=x$, then $\dot{F}=x^2\dot{x}$, FIG. 173.
and $F=\frac{x^3}{3}$; also $\dot{M}=x\dot{x}$, and $M=\frac{x^2}{2}$;
whence $\frac{F}{M}=\frac{2}{3}x=CO$.

Ex. 2.

In a Parallelogram where the Axis of Motion is in 174.
the Plane of the Figure, $CB=x$, $BD=b$, then $\dot{F}=bx^2\dot{x}$, and $F=\frac{bx^3}{3}$; also $\dot{M}=bx\dot{x}$, and $M=\frac{bx^2}{2}$; whence $\frac{F}{M}=\frac{2}{3}x=CO$.

Ex. 3.

Let AD be the Arch of a Circle, the Center C the 175.
Point of Suspension, and the Axis of Motion perpendicular to it's Plane; let Arch $ABD=s$, Cord $AD=c$, $CB=r$. Then $F=rrs$; and by Ex. 3. Prob. XVII, $d=\frac{rc}{s}$; therefore $\frac{F}{dB}=\frac{F}{ds}=\frac{rs}{c}$.

Ex. 4.

Let AD be a right Line, the Axis of Motion per- 176.
pendicular to the Plane passing through it. $CB=d$, $BA=y$, then $\dot{F}=\overline{dd+yy}\times 2\dot{y}$, and $F=2ddy+\frac{2}{3}y^3$; also $M=2dy$; whence $\frac{F}{M}=\frac{2ddy+\frac{2}{3}y^3}{2dy}=d+\frac{yy}{3d}$.

Ex. 5.

For the Periphery of a Circle, let $CD=d$, Radius 177.
$AD=r$, Circumference $=c$. If the Axis of Motion be perpendicular to it's Plane, then $G=rrc$, and $d+\frac{G}{dB}=d+\frac{rrc}{dc}=d+\frac{rr}{d}$.

But if the Axis of Motion be parallel to the Plane of the Circle, let $DE=z$, $PQ=s$, then $\dot{G}=z^2\dot{s}$

FIG. $= \frac{4rz^2\dot{z}}{\sqrt{rr-zz}}$ and the Fluent $G = \frac{rrc}{2}$, for the whole Circle: Therefore $d + \frac{G}{dB} = d + \frac{rr}{2d}$.

Ex. 6.

177. *For the Plane of the Circle*, the Axis of Motion perpendicular to it's Plane, $DE = z$, Circumference at $E = \frac{cz}{r}$, then $\dot{G} = \frac{cz^3\dot{z}}{r}$, and $G = \frac{cz^4}{4r} = \frac{cr^3}{4}$. Then $d + \frac{G}{dB} = d + \frac{cr^3}{4dB} = d + \frac{rr}{2d}$.

And if the Axis of Motion be parallel to the Plane of it, $\dot{G} = 4z^2\dot{z}\sqrt{rr-zz}$, whence $G = \frac{cr^3}{8}$; and $d + \frac{G}{dB} = d + \frac{cr^3}{8dB} = d + \frac{rr}{4d}$,

Ex. 7.

178. *For the Periphery of the Circle*, parallel to the Horizon, let the Axis of Motion be parallel to ED, Radius $DB = r$, $BA = v$, $DL = z$, $CD = d$, then $\dot{G} = 4zz\dot{v} = \frac{4rzz\dot{z}}{\sqrt{rr-zz}}$, and the whole Fluent $G = \frac{rrc}{2}$; then $d + \frac{G}{dB} = d + \frac{rrc}{2dc} = d + \frac{rr}{2d}$.

Ex. 8.

178. *For the Plane of the Circle*, whose Point of Suspension is in CD perpendicular to it's Plane. Let $AL = x$, the rest as before, then $\dot{v} = x\dot{z}$, and $\dot{G} = 4xz^2\dot{z} = 4z^2\dot{z} \times \sqrt{rr-zz}$, and the whole Fluent $G = \frac{cr^3}{8}$; and $d + \frac{G}{dB} = d + \frac{rr}{4d}$.

Ex. 9.

179. *In an Isoceles Triangle*, where the Axis of Motion is parallel to the Base, let $CD = a$, $CA = x$, $BE = f$, then

FIG.

then $IK = \frac{fx}{a}$, and $\dot{F} = \frac{fx^3\dot{x}}{a}$; whence $F = \frac{fx^4}{4a}$; also $\dot{M} = \frac{fx^2\dot{x}}{a}$, and $M = \frac{fx^3}{3a}$; therefore $\frac{F}{M} = \frac{3}{4}x = \frac{3}{4}a$.

If the Axis at C be perpendicular to the Plane of the Triangle, let $AQ = v$, then $\ddot{F} = \overline{xx+vv} \times \dot{x}\dot{v}$, and $\dot{F} = x^2\dot{x}v + \frac{v^3\dot{x}}{3} =$ (because $v = \frac{fx}{a}$) $\frac{fx^3\dot{x}}{a} + \frac{f^3x^3\dot{x}}{a^3}$; whence $F = \frac{fx^4}{4a} + \frac{f^3x^4}{4a^3} = \frac{fa^3+f^3a}{4}$; therefore $\frac{F}{M} = \frac{3}{4} \times \frac{aa+ff}{a}$.

Ex. 10.

In the Parabola CAF, $AC = x$, $AF = y$, $AQ = v$, $ax = yy$; let the Axis in C be parallel to AF, then $\dot{F} = yx^2\dot{x}, = x^2\dot{x}\sqrt{ax}$, and $F = \frac{2}{7}x^3\sqrt{ax}$; and $\dot{M} = yx\dot{x} = x\dot{x}\sqrt{ax}$, and $M = \frac{2}{5}x^2\sqrt{ax}$. Therefore $\frac{F}{M} = \frac{5}{7}x$. 180.

If the Axis be perpendicular to it's Plane; then $\ddot{F} = \overline{xx+vv} \times \dot{x}\dot{v}$, and $\dot{F} = x^2v\dot{x} + \frac{v^3\dot{x}}{3} = yx^2\dot{x} + \frac{y^3\dot{x}}{3} = x^2\dot{x}\sqrt{ax} + \frac{ax\dot{x}\sqrt{ax}}{3}$; whence $F = \frac{2}{7}x^3\sqrt{ax} + \frac{2}{15}ax^2\sqrt{ax}$. Therefore $\frac{F}{M} = \frac{5}{7}x + \frac{1}{3}a$.

Ex. 11.

Let AB be the Surface of a Sphere, $AB = s$, $BE = z$, Radius $AD = r$, $c =$ Circumference. Then Circumference of $BE = \frac{cz}{r}$, and $\dot{G} = \frac{cz^3\dot{s}}{r} = \frac{cz^3}{r} \times \frac{r\dot{z}}{\sqrt{rr-zz}}$; whence $G = \frac{2cr^3}{3} - \overline{\frac{2cr^2}{c} - \frac{czz}{3}}\sqrt{rr-zz} = \frac{2cr^3}{3}$. And $B = rc$; therefore $d + \frac{G}{dB} = d + \frac{2rr}{3d}$. 181.

Ex. 12.

Ex. 12.

FIG. 182. *Let ABQD be a Parallelepipidon*, the Axis of Motion perpendicular to the Plane $ABQD$; let $AB=2a$, $AD=2b$, Breadth $=c$, $GS=x$, $SZ=y$; then $\ddot{G}=\overline{xx+yy}\times 4c\dot{x}\dot{y}$, and $\dot{G}=4cx^2y\dot{x}+\frac{4cy^3\dot{x}}{3}=4cbx^2\dot{x}+\frac{4}{3}cb^3\dot{x}$: Therefore $G=\frac{4}{3}cbx^3+\frac{4}{3}cb^3x=\frac{4}{3}cba^3+\frac{4}{3}cb^3a$; also $B=4acb$: Therefore $d+\frac{G}{dB}=d+\frac{aa+bb}{3d}$.

Ex. 13.

183. *In a Cylinder* let $CA=x$, $AD=r$, $Ae=y$, $CH=a$, the Axis of Motion parallel to AB; then $\ddot{F}=\overline{xx+yy}\times 4\dot{x}\dot{y}\sqrt{rr-yy}$, and the whole Fluent $\dot{F}=\frac{rcx^2\dot{x}}{2}+\frac{r^3c\dot{x}}{8}$. And $F=\frac{rcx^3}{6}+\frac{r^3cx}{8}$; which corrected gives $F=\frac{rcx^3-rca^3}{6}+\frac{r^3cx-r^3ca}{8}$, and $B=\overline{x-a}\times\frac{rc}{4}$, then will $\frac{F}{dB}=\frac{2x^3-2a^3}{3x^2-3aa}+\frac{rrx-rra}{2x^2-2aa}=\frac{4xx+4ax+4aa+3rr}{6x+6a}$.

Ex. 14.

184. *Let CKB be a Pyramid*, whose Base is a Parallelogram, and Axis of Motion in C perpendicular to the Plane CEF; let it's Altitude $=a$, $AB=f$, $AD=c$, $CH=x$, $HL=y$, then $KI=\frac{cx}{a}$, and $EF=\frac{fx}{a}$, and $\ddot{F}=\overline{xx+yy}\times\frac{cx\dot{x}\dot{y}}{a}$; and $\dot{F}=\frac{cx^3\dot{x}}{a}y+\frac{cx\dot{x}y^3}{3a}=\frac{cfx^4\dot{x}}{aa}+\frac{cf^3x^4\dot{x}}{12a^4}$. And $F=\frac{cfx^5}{5aa}+\frac{cf^3x^5}{60a^4}$. Also $\dot{M}=\frac{cfx^3\dot{x}}{aa}$, and $M=\frac{cfx^4}{4aa}$; whence $\frac{F}{M}=\frac{4}{5}x+\frac{ff}{15aa}x=\frac{12aa+ff}{15a}$.

Ex. 15.

Ex. 15.

In a right Cone, let $CA=g$, Altitude $=a$, Radius of the Base $=f$, $AB=x$, $BI=z$, $c=3,1416$, then FIG. 185.
BE or $BD=\frac{fx}{a}$, $IE=\sqrt{\frac{ffxx}{aa}-zz}$; then $\ddot{F}=\overline{g+x}^2+zz\times 4\dot{x}\dot{z}\times\sqrt{\frac{ffxx}{aa}-zz}$, and $\dot{F}=\overline{g+x}^2\times\frac{cffx^2\dot{x}}{aa}+\frac{cf^4x^4\dot{x}}{4a^4}$, consequently $F=:\frac{1}{3}ggcx^3+\frac{1}{2}gcx^4+\frac{1}{5}cx^5+\frac{cffx^5}{20aa}:\times\frac{ff}{aa}$; also $\dot{M}=\overline{g+x}\times\frac{cffx^2\dot{x}}{aa}$, and $M=:\frac{1}{3}gcx^3+\frac{1}{4}cx^4:\times\frac{ff}{aa}$; whence $\frac{F}{M}=\frac{20gg+30ga+12aa+3ff}{20g+15a}$.

Ex. 16.

For the Sphere AS, draw the Diameter AS parallel 181.
to the Axis of Motion at C; let $AD=r$, $DE=x$, $Ef=z$, $EB=y$, $c=3.1416$. Then $\ddot{G}=2cz^3\dot{x}\dot{z}$, and $\dot{G}=\frac{cz^4\dot{x}}{2}=\frac{1}{2}cy^4\dot{x}=\frac{c\dot{x}}{2}\times\overline{r^4-2rrxx+x^4}$; whence $G=\frac{1}{2}cr^4x-\frac{1}{3}cr^2x^3+\frac{1}{10}cx^5$. Also $cyy\dot{x}=$ Fluxion of the Solid $=c\dot{x}\times\overline{rr-xx}$. Therefore the Solid $=crrx-\frac{1}{3}cx^3=B$. Whence $\frac{G}{dB}=\frac{15r^4-10r^2x^2+3x^4}{30rr-10xx:\times d}$; and for the whole Sphere $\frac{G}{dB}=\frac{2rr}{5d}=DO$; and $CO=d+\frac{2rr}{5d}$.

Ex. 17.

Let AD be a Paraboloid; $CA=a$, $AB=x$, $BD=y$, 186.
$BI=z$, $c=3.1416$, $rx=yy$. Then $\ddot{F}=:\overline{a+x}^2+zz:\times 4\dot{x}\dot{z}\times\sqrt{yy-zz}$; therefore $\dot{F}=\overline{a+x}^2\times cyy\dot{x}+\frac{cy^4}{4}\dot{x}=\overline{a+x}^2\times crx\dot{x}+\frac{1}{4}cr^2x^2\dot{x}$; whence $F=\frac{1}{2}caarxx+\frac{2}{3}carx^3$
+

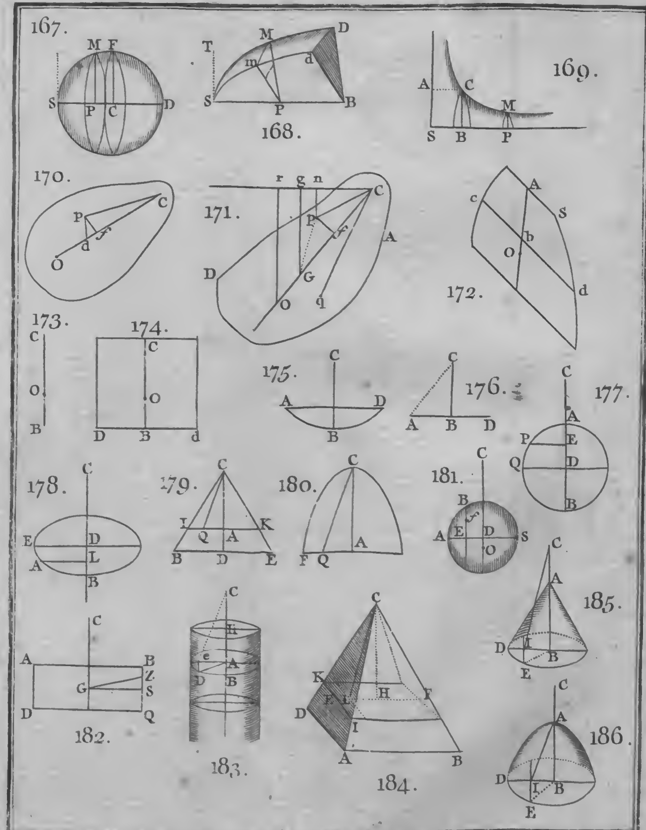
FIG. $+\frac{1}{4}crx^4+\frac{1}{12}crrx^3$. Alſo $\dot{M}=\overline{a+x}\times cy^2\dot{x}=\overline{a+x}\times crx\dot{x}$, and $M=\frac{1}{2}carx^2+\frac{1}{3}crx^3$. Therefore $\frac{F}{M}=\frac{6a^2+8ax+rx+3xx}{6a+4x}$.

PROB. XIX.

187. *To find the Law of centripetal Force requiſite to cauſe a Body to move in a given Curve BF.*

Let B be the Place of the Body moving in the Orbit BF by a Force directed to the given Point C. Draw the Tangent BY, and the Radius CB, CQ, infinitely near each other, QR parallel to CB, and CY, QN Perpendiculars to BY. Let the Diſtance $CB=D$, Perpendicular $CY=P$, then the infinitely ſmall Line QR will be as the Force and Square of the Time conjunctly, that is as the Force and Square of the Area CBQ; therefore the Force is as $\frac{QR}{\overline{CBQ}^2}$ or $\frac{QR}{QB^2\times P^2}$, that is (becauſe $P:D::Qn:QR=\frac{Qn\times D}{P}$) as $\frac{Qn\times D}{QB^2\times P^3}$. But $\frac{QB^2}{2Qn}=$ Radius of Curvature in the Point B, and the ſame Radius is alſo $\frac{D\dot{D}}{\dot{P}}$ by Prob. V. wherefore the Force is as $\frac{\dot{P}}{2P^3\dot{D}}$, that is (ſuppoſing $\dot{D}$ to be given) as the Fluxion of $\frac{-1}{PP}$.

Therefore to find the Law of centripetal Force, let $D=$ Diſtance from the Center of Force, $P=$ Perpendicular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dicular on the Tangent: Compute the Value of P in Terms of D, by the Nature of the Curve; then find the Fluxion of $\frac{-1}{PP}$, making $\dot{D}=1$, and then expunge all Quantities as far as possible, except D, and you will have F, the Law of centripetal Force required. FIG.

Ex. 1.

Let C the Center of Force be in the Circumference of the Circle CBD; Diameter $CD=2r$, the Triangles 188.
CBD and $CB\Upsilon$ are similar, whence $P=\frac{DD}{2r}$, therefore $\frac{-1}{PP}=\frac{-4rr}{D^4}$, therefore $F \propto \frac{16r^2\dot{D}}{D^5} \propto \frac{1}{D^5}$; that is, the Force is reciprocally as the fifth Power of D.

Ex. 2.

Let DB be a Circle; and C at an infinite Distance, 189.
$AE=x$, $EB=y$, $AD=r$, then $P=\frac{Dy}{r}$, and $\frac{-1}{PP}=\frac{-rr}{DD}y^{-2}$, whose Fluxion (because D is a standing Quantity) is $\frac{2r^2}{DD}y^{-3}\dot{y}$, therefore $F \propto \frac{2r^2}{D^2}y^{-3}$ or $F \propto \frac{1}{y^3}$.

Ex. 3.

Let BF be an Ellipsis; C the Focus, E the Center; 190.
draw BE and it's Conjugate AE, and let the transverse Axis $=2r$, Conjugate $=2c$; then by the Property of the Ellipsis $2rD-DD$ the Rectangle of the focal Distances from B, is $=AE^2$, also AE or $\sqrt{2rD-DD}:c::D:P=\frac{cD}{\sqrt{2rD-DD}}$; and $\frac{-1}{PP}=\frac{-2rD+DD}{ccDD}$

 =

FIG. $= \frac{-2r}{ccD} + \frac{1}{cc}$, whose Fluxion is $\frac{2r\dot{D}}{ccDD}$, therefore $F \propto \frac{2r}{ccDD}$ or $\frac{1}{DD}$.

Ex. 4.

191: *Let C be the Center of the Ellipsis*; then by the Nature of the Figure $AC^2 + CB^2 = rr + cc$, and $AC = \sqrt{rr+cc-DD}$; also $AC : c :: r : P = \frac{cr}{\sqrt{rr+cc-DD}}$; therefore $-\frac{1}{PP} = \frac{-rr-cc+DD}{ccrr}$, whose Fluxion is $\frac{2D\dot{D}}{ccrr}$; therefore $F \propto \frac{2D}{ccrr} \propto D$. That is, the Force is as the Distance.

Ex. 5.

192. *Let BA be an Hyperbola, C* the Focus. Proceeding as in the Ellipsis, we shall find $P = \frac{cD}{\sqrt{2rD+DD}}$, and $-\frac{1}{P^2} = \frac{-2rD-DD}{ccDD} = -\frac{2r}{ccD} - \frac{1}{cc}$, whose Fluxion is $\frac{2r\dot{D}}{ccDD}$, therefore $F \propto \frac{2r}{ccDD}$ or $F \propto \frac{1}{DD}$.

And after the same Manner if the Force be in the other Focus, there will be found $P = \frac{cD}{\sqrt{DD-2rD}}$, and the Force $F \propto \frac{-2r}{ccDD} \propto \frac{-1}{DD}$, and is therefore a centrifugal Force.

Ex. 6.

193. *Let AB be an Hyperbola, C* the Center, Transverse $= 2r$, Conjugate $= 2c$, $b =$ half the Conjugate belonging to CB. Then by the Property of the Hyperbola

bb

$bb - DD = rr - cc$, and $b = \sqrt{rr - cc + DD}$; and FIG.
$b : c :: r : P = \frac{cr}{\sqrt{rr - cc + DD}}$. Consequently $-\frac{1}{PP} = \frac{-rr + cc - DD}{ccrr}$, whose Fluxion is $\frac{-2D\dot{D}}{ccrr}$, therefore $F \propto \frac{-2D}{ccrr} \propto -D$, and is therefore a centrifugal Force.

Ex. 7.

Let AB be a Parabola, C the Focus, r = Latus Rectum; by the Property of the Figure $P = \sqrt{\frac{rD}{4}}$, whence $-\frac{1}{PP} = \frac{-4}{rD}$, whose Fluxion is $\frac{4\dot{D}}{rDD}$; therefore $F \propto \frac{1}{DD}$. 194.

Ex. 8.

195. *Let AB be a Parabola, C* the Center of Force at an infinite Distance in the Axis. $AD = x$, $DB = y$, $rx = yy$, then $P = \frac{Dy}{\sqrt{4xx + ax}}$, and $\frac{-1}{PP} = \frac{-4xx - ax}{DDyy} = \frac{-4x}{aDD} - \frac{1}{DD}$, whose Fluxion is $\frac{-4\dot{x}}{aDD} + \frac{8x\dot{D}}{aD^3} + \frac{\dot{D}}{D^3} =$ (because $-\dot{x} = \dot{D}$): $\frac{4D}{aD^3} + \frac{8x + a}{aD^3} : \times \dot{D}$; therefore $F \propto \frac{4D + 8x + a}{aD^3}$, or (because D is infinite) $\propto \frac{4D}{aD^3}$, or $F \propto \frac{4}{aD^2}$ a given Quantity.

Ex. 9.

196. *Let C be the Vertex of the Parabola*, $CA = x$, $AB = y$, $ax = yy$; by similar Triangles $TB : AB :: TC$ or $x : CY$ or $P = \frac{xy}{\sqrt{4xx + yy}}$; whence $-\frac{1}{PP} = -\frac{4xx + yy}{xxyy}$.

FIG. $\frac{4xx+yy}{xxyy} = \frac{-4}{ax} - \frac{1}{xx}$; whose Fluxion is $\frac{4\dot{x}}{ax^2} + \frac{2\dot{x}}{x^3}$. But $DD = xx + ax$, and thence $\dot{x} = \frac{2D\dot{D}}{a+2x}$, therefore $F \propto \frac{4x+2a}{ax^3} \times \frac{2D\dot{D}}{2x+a} = \frac{4D\dot{D}}{ax^3}$, or $F \propto \frac{D}{x^3}$.

Ex. 10.

197. *Let VAR be an Ellipsis, C the Focus; and let the Curve VBQ be formed from the Ellipsis, thus; take CA = Cb, and the Angle VCA to the Angle VCb, as m to n, for all the Points of the Curve. To find the Law of centripetal Force of a body moving in the Curve VBQ.*

Let the Transverse of the Ellipsis $= 2r$, Conjugate $= 2c$. By the Property of the Ellipsis the Perpendicular $Cp = \frac{cD}{\sqrt{2rD - DD}}$, also $tb = TA$, and $\angle BCb = \frac{n}{m} ACo$. And by similar Triangles $\frac{P \times tb}{BY} = tB = \frac{n}{m} TO = \frac{\frac{n}{m} Cp \times TA}{pA}$, that is $\frac{P}{\sqrt{DD-PP}} = \frac{nc}{m\sqrt{2rD - DD - cc}}$; whence by Reduction $PP = \frac{n^2c^2D^2}{2rm^2D - m^2D^2 - m^2c^2 + n^2c^2}$; whence $\frac{-1}{PP} = \frac{-2rm^2}{n^2c^2D} + \frac{m^2}{n^2c^2} - \frac{n^2c^2 - m^2c^2}{n^2c^2D^2}$; whose Fluxion is $\frac{2rm^2\dot{D}}{n^2c^2D^2} + \frac{2n^2 - 2m^2}{n^2c^2D^3} c^2\dot{D}$, therefore $F \propto \frac{m^2}{DD} + \frac{n^2 - m^2}{rD^3} cc$. In the same Manner if C were the Center of the Ellipsis, it might be proved that the Force is as $\frac{m^2D}{rr} + \frac{nn - mm}{D^3} cc$.

Ex. 11.

Ex. 11.

Suppose AB to be the Logarithmic Spiral; since the Angle CBY is always given, there is given the Ratio of CB to CY suppose as m to n. Then $P = \frac{n}{m} D$; and $\frac{-1}{PP} = \frac{-mm}{nnDD}$, whose Fluxion is $\frac{2m^2\dot{D}}{nnD^3}$; therefore $F \propto \frac{1}{D^3}$. FIG. 198.

Ex. 12.

Let CB be the hyperbolic Spiral; draw CT perpendicular to CB, and let $CB=y$, the given Subtangent $CT=a$. By similar Triangles $\sqrt{aa+yy} : y :: a : P = \frac{ay}{\sqrt{aa+yy}}$; and $-\frac{1}{PP} = -\frac{1}{yy} - \frac{1}{aa}$, whose Fluxion is $\frac{2\dot{y}}{y^3} = \frac{2\dot{D}}{D^3}$, therefore $F \propto \frac{1}{y^3} \propto \frac{1}{D^3}$. 199.

PROB. XX.

The Nature of the Curve ABD forming an Arch being given; to find the Nature of the Curve RST bounding the Top of the Wall ATRD supported by that Arch; by the Pressure or Weight of which Wall, all the Parts of the Arch are kept in Equilibrio without falling. 200.

1. Let several equal right Lines AB, BC, CD, &c, placed in a vertical Plane, be movable round the Angles A, B, C, D, &c, whilst the Points A, G, at the Base remain fixed and immovable. Through B, 201.

FIG. 201. B, C, D, &c, draw the Lines Bi, Cm, Dp, &c, perpendicular to the Horizon; and complete the Parallelogram $Bhik$, and make $Cl = Bk$, and complete the Parallelogram $Clmn$. In like Manner make $Do = Cn$ or lm, $Er = op$, $Ft = rs$, and complete all the Parallelograms in the Figure as at firſt.

2. Let ſeveral Weights which are to one another as the Lines Bi, Cm, Dp, &c, lye reſpectively on the Points B, C, D, &c. Now the Force Bi, is equivalent to Bh, Bk, acting in the Directions BA, BC; the Force Bh is deſtroyed by the Reſiſtance of the Point A; but Bk endeavours to move the Point B towards C. In like Manner the Force Cm is equivalent to Cl and Cn; the Force Dp to Do, op, &c. Now the Forces Bk (acting towards C) and Cl (acting towards B) being equal by Conſtruction deſtroy one another. In like manner the Forces Cn, Do; Dq and Er; Ev and Ft, &c, deſtroy one another; and the Point G being fixed, it is manifeſt the Figure $ABCD$, &c, will not be moved by the incumbent Weights Bi, Cm, Dp, &c, but all it's Parts will remain in Equilibrio.

3. The Force Bh : Force Bk or Cl :: Sine $\angle\, hiB$ or $iBC : S.\,\angle\, ABi :: \frac{1}{S.\,ABi} : \frac{1}{S.iBC}$ or $\frac{1}{S.mCB}$. Likewiſe Force Cl : Force Cn or Do :: $S.mCD$ or $pDC : S.mCB :: \frac{1}{S.mCB} : \frac{1}{S.mCD}$ or $\frac{1}{S.pDC}$, and ſo on; whence it is plain in general, that any Force Cl is as $\frac{1}{S.\,\angle\, mCB}$. Now ſince $Cm = \frac{S.\,Clm \times Cl}{S.Cml} = \frac{S.\,BCx \times Cl}{S.\,mCD}$; therefore the Force $Cm \propto \frac{S.\,BCx}{S.mCB \times S.mCD}$.

4. Now let the Number of the Lines AB, BC, CD, &c, be increaſed and their Lengths diminiſhed *ad infinitum*, that the Figure may obtain the Form of a Curve,

a Curve, and the Pressure will then act on all Parts of it; and the Angle BCx will then become the Angle of Contact, and the Sines of mCB and mCD become equal to the Sine of mCx: Therefore drawing the Tangent An (Fig. 200.) the Pressure on any Point A to preserve the Equilibrium will be as the Angle of Contact at A directly and the Square of the Sine of the Angle mAn reciprocally. But the Angle of Contact is as the Curvature, or reciprocally as the Radius of Curvature. Therefore the Pressure is reciprocally as that Radius and the Square of the Sine of that Angle mAn. FIG. 200.

5. Let $BC=x$, $AC=y$, $AB=z$. Radius of Curvature in $A=R$. Then if $\dot{z}$ be given, $S. \angle TAN \propto \dot{y}$, and $\frac{1}{S.TAN} \propto \frac{1}{\dot{y}}$. Then the Weight or Pressure on $A = AT \times \dot{y}$, and that (as has been proved) is as $\frac{1}{R \times S.TAN^2} \propto \frac{1}{R\dot{y}^2}$, therefore $AT \propto \frac{1}{R\dot{y}^3}$, or in general $AT \propto \frac{\dot{z}^3}{R\dot{y}^3}$. But when $\dot{x}$ is given $R = \frac{\dot{z}^3}{-\dot{x}\ddot{y}}$; or if $\dot{y}$ be given $R = \frac{\dot{z}^3}{\dot{y}\ddot{x}}$; therefore $AT \propto \frac{-\dot{x}\ddot{y}}{\dot{y}^3}$ if $\dot{x}$ be given; or $AT \propto \frac{\ddot{x}}{\dot{y}^2}$, if $\dot{y}$ be given.

Wherefore to find the Curve ST, let $BC=x$, $AC=y$, $AB=z$; then by the Nature of the Curve AB, compute $\frac{-\dot{x}\ddot{y}}{\dot{y}^3}$ if $\dot{x}$ be given, or $\frac{\ddot{x}}{\dot{y}^2}$ if $\dot{y}$ be given, and take AT proportional thereto. And x may be expunged out of the Value of AT, by Help of the given Line BS, and thence the Nature of the Curve ST will be known.

Ex. 1.

Let BA be a Circle, Radius $AR=r$, $BC=x$, $AC=y$, $BS=a$, $\dot{x}$ given. Then $y=\sqrt{2rx-xx}$, $\dot{y}=\frac{r-x}{\sqrt{2rx-xx}}\dot{x}$, 202.

$\ddot{y}=$

FIG. $\ddot{y} = \frac{-rr\dot{x}^2}{\overline{2rx-xx}|^{\frac{3}{2}}}$; then will $\frac{-\dot{x}\ddot{y}}{\dot{y}^3} = \frac{rr\dot{x}^3}{\overline{2rx-xx}|^{\frac{3}{2}}}$ $\times \frac{\overline{2rx-xx}|^{\frac{3}{2}}}{\overline{r-x}|^3 \times \dot{x}^3} = \frac{rr}{\overline{r-x}|^3}$; therefore $AT \propto \frac{rr}{\overline{r-x}|^3}$, or $AT = \frac{r^3a}{\overline{r-x}|^3}$. And expunging x, $SQ = (AT - CS =) \frac{r^3a}{\overline{rr-yy}|^{\frac{3}{2}}} - a - r + \sqrt{rr-yy}$, for the Nature of the Curve ST.

Hence the Curve ST runs upwards *ad infinitum*, and the Perpendicular DE is an Affymptote to the Curve.

SCHOLIUM.

If ST had been a right Line, then the Point A would be preffed with too little Weight, and B with too much: And hence appears the Reafon why circular Arches commonly break about the Top, by being loaded there with more Weight than their due Proportion.

Ex. 2.

202. *Let the Curve DAB be an Ellipfis*; $BR = r$, $DR = c$, $CB = x$, $AC = y$, $BS = a$, $\dot{x}$ given. Then $y = \frac{c}{r}\sqrt{2rx-xx}$, $\dot{y} = \frac{c\dot{x}}{r} \times \frac{r-x}{\sqrt{2rx-xx}}$, $\ddot{y} = \frac{-cr\dot{x}^2}{\overline{2rx-xx}|^{\frac{3}{2}}}$: whence $\frac{-\dot{x}\ddot{y}}{\dot{y}^3} = \frac{r^4}{cc \times \overline{r-x}|^3}$, which is as AT; therefore $AT = \frac{r^3a}{\overline{r-x}|^3}$. Then $SQ = \frac{r^3a}{\overline{r-x}|^3} - a - x$ $=$ (expunging x) $\frac{c^3a}{\overline{cc-yy}|^{\frac{3}{2}}} - a - r + \frac{r}{c}\sqrt{cc-yy}$, for the Nature of the Curve ST, being of the fame Kind with the Foregoing.

Ex. 3.

Let AB be a Parabola, $BC=x$, $CA=y$, $SB=a$, $rx=yy$, $\dot{y}$ given. Then $\dot{x}=\frac{2y\dot{y}}{r}$, $\ddot{x}=\frac{2\dot{y}^2}{r}$, and $\frac{\ddot{x}}{\dot{y}^2}=\frac{2}{r}$, a given Quantity, therefore AT is every where the same, and ST is the same Parabola placed in a higher Position. FIG. 203.

Ex. 4.

Let BA be an Hyperbola, Transverse $=2r$, Conjugate $=2c$, $BC=x$, $CA=y$, $BS=a$, $\dot{y}$ given. Then $2rx+xx=\frac{rryy}{cc}$, whence $r+x=\frac{r}{c}\sqrt{cc+yy}$, $\dot{x}=\frac{r\dot{y}}{c}\times\frac{y}{\sqrt{cc+yy}}$, $\ddot{x}=\frac{rc\dot{y}^2}{\overline{cc+yy}|^{\frac{3}{2}}}$; wherefore $AT \propto$ $\left(\frac{\ddot{x}}{\dot{y}^2}=\right)\frac{rc}{\overline{cc+yy}|^{\frac{3}{2}}}$, and $AT=\frac{c^3a}{\overline{cc+yy}|^{\frac{3}{2}}}$. 203.

Hence the Curve ST continually approaches nearer and nearer the Hyperbola: And $SQ=(a+x-AT=)$ $a-r+\frac{r}{c}\sqrt{cc+yy}-\frac{c^3a}{\overline{cc+yy}|^{\frac{3}{2}}}$, expresses the Nature of the Curve ST.

Ex. 5.

Let BA be a Cycloid, $BC=x$, $CA=y$, $BD=s$, $BV=a$, $\dot{x}$ given. By the Property of the Curve $y=s+\sqrt{ax-xx}$, but $\dot{s}=\frac{a\dot{x}}{2\sqrt{ax-xx}}$, then $\dot{y}=\frac{2a\dot{x}-2x\dot{x}}{2\sqrt{ax-xx}}=\dot{x}\sqrt{\frac{a-x}{x}}$, and $\ddot{y}=\frac{-a\dot{x}^2}{2x\sqrt{ax-xx}}$; therefore $\frac{-\dot{x}\ddot{y}}{\dot{y}^3}=\frac{r}{2\times\overline{a-x}|^2}$; Therefore $AT \propto$ $\frac{1}{\overline{a-x}|^2}\propto\frac{1}{\overline{CV}|^2}$. 204.

Ex. 6.

FIG. 203. *Let BA be the Catenary*, $BC=x$, $CA=y$, $BA=z$, $BS=a$, $\dot{x}$ given; then $z=\sqrt{2rx+xx}$, $\dot{z}=\frac{r+x}{\sqrt{2rx+xx}}\dot{x}$, therefore $\dot{y}=(\sqrt{\dot{z}^2-\dot{x}^2}=)\ \frac{r\dot{x}}{\sqrt{2rx+xx}}$, and $\ddot{y}=\frac{-r\dot{x}^2\times\overline{r+x}}{\overline{2rx+xx}|^{\frac{3}{2}}}$; whence $\frac{-\dot{x}\ddot{y}}{\dot{y}^3}=\frac{r+x}{rr}$, therefore AT is as $r+x$, or $AT=\frac{a}{r}\times\overline{r+x}$. Hence,

1. If $a=r$, then $AT=a+x=CS$, and then ST is a right Line paſſing through S.

2. If BS is very ſmall, draw At perpendicular to the Curve, and by ſimilar Triangles $(\dot{z}=)\ \frac{\overline{r+x}\times\dot{x}}{\sqrt{2rx+xx}}$: $(\dot{y}=)\ \frac{r\dot{x}}{\sqrt{2rx+xx}}$ :: $(AT=)\ \frac{a}{r}\times\overline{r+x}$: $At=a=BS$; therefore the Arch is of the ſame Thickneſs every where: Conſequently a heavy flexible Line put into this Figure would ſupport it ſelf.

3. For the Nature of the Curve ST, we have $SQ=(a+x-AT=)\ x-\frac{ax}{r}$. Therefore if a is leſſer than r the Curve is concave towards B, and if a is greater than r, it is convex towards B.

Ex. 7.

205. *Let AB be the logarithmic Curve, GD it's Aſſymptote*, $BS=a$, $BD=r$, Subtangent $GE=t$, $BC=x$, $CA=y$, $AG=r+x$, then by the Property of the Curve $\dot{y}=\frac{t\dot{x}}{r+x}$, and if $\dot{x}$ be given, $\ddot{y}=\frac{-t\dot{x}^2}{\overline{r+x}|^2}$, whence $\frac{-\dot{x}\ddot{y}}{\dot{y}^3}=\frac{t\dot{x}^3\times\overline{r+x}|^3}{\overline{r+x}|^2\times t^3\dot{x}^3}=\frac{r+x}{tt}$, whence $AT \propto \overline{r+x}$, or $AT=\frac{a}{r}\times\overline{r+x}$, the ſame as in the laſt Example.

For

For the Nature of the Curve, $SQ = (a + x - AT$ FIG.
$=)\ x - \frac{ax}{r}$, therefore if $a = r$, then ST becomes the Assymtote DG, and if a be less than r, the Curve ST is concave; but if a is greater than r, it is convex towards B.

Or thus for the Nature of the Curve, $GT = (r + x - AT =)\ r - a + \frac{r-a}{r}x$, for the Relation of ST to the Assymptote GD.

Ex. 8.

Let AB *be the Cissoid*, whose Equation is $axx - yxx$ 206.
$= y^3$, in Fluxions $2ax\dot{x} - 2yx\dot{x} - x^2\dot{y} = 3y^2\dot{y}$; this again in Fluxions (making $\ddot{x} = 0$), $2a\dot{x}^2 - 2y\dot{x}^2 - 2x\dot{x}\dot{y} - 2x\dot{x}\dot{y} - x^2\ddot{y} = 6y\dot{y}^2 + 3y^2\ddot{y}$; from the former Equation $\dot{y} = \frac{2ax - 2yx}{3yy + xx}\dot{x}$, and from the latter, $\ddot{y} = \frac{\overline{2a - 2y} \times \dot{x}^2 - 4x\dot{x}\dot{y} - 6y\dot{y}^2}{3yy + xx}$. Now if $a = 10$, $\dot{x} = 1$; suppose $y = 2$, $x = 1$, to find AT; here $\dot{y} = \frac{16}{13}$, $\ddot{y} = -,5462$; whence $\frac{-x\ddot{y}}{\dot{y}^3} = ,293$ for the Value of AT. And to find it in the Vertex where x and y are 0, and $\dot{y}$ infinitely greater than $\dot{x}$; we shall have $axx = y^3$, $\dot{y} = \frac{2ax}{3yy} = \frac{2}{3}\sqrt{\frac{a}{y}}$, $\ddot{y} = \frac{2a - 4x\dot{y} - 6y\dot{y}^2}{3yy}$
$= \frac{-2a}{9yy}$; whence $\frac{-x\ddot{y}}{\dot{y}^3} = \frac{3}{4\sqrt{ay}} =$ Infinity.

More generally thus:

Since $axx - yxx = y^3$, in Fluxions $2ax\dot{x} - 2yx\dot{x}$
$-x^2\dot{y} = 3y^2\dot{y}$, whence $\dot{x} = \frac{3xy^2 + x^3}{2y^3}\dot{y}$, this again in Fluxions and reduced (making $\dot{y}$ invariable), $\ddot{x} =$
$\frac{3}{2} \times \frac{\overline{y^3 + x^2y} \times \dot{x} - \overline{xy\dot{y} - x^3} \times \dot{y}}{y^4} \times \dot{y} =$ (expunging xx)

 $\frac{3}{2}\dot{y}$

FIG. $\frac{3}{2}\dot{y} \times \frac{a y\dot{x} - a x\dot{y}}{yy \times \overline{a-y}} =$ (expunging $\dot{x}$) $\frac{3aax\dot{y}^2}{4yy \times \overline{a-y}^2} =$ (expunging x) $\frac{3aa\dot{y}^2}{\overline{a-y}^2 \times 4\sqrt{ay-yy}}$; whence $\frac{\ddot{x}}{\dot{y}^2} = \frac{3aa}{\overline{a-y}^2 \times 4\sqrt{ay-yy}}$: Therefore $AT \propto \frac{1}{\sqrt{y} \times \overline{a-y}^{2\frac{1}{2}}}$.

PROB. XXI.

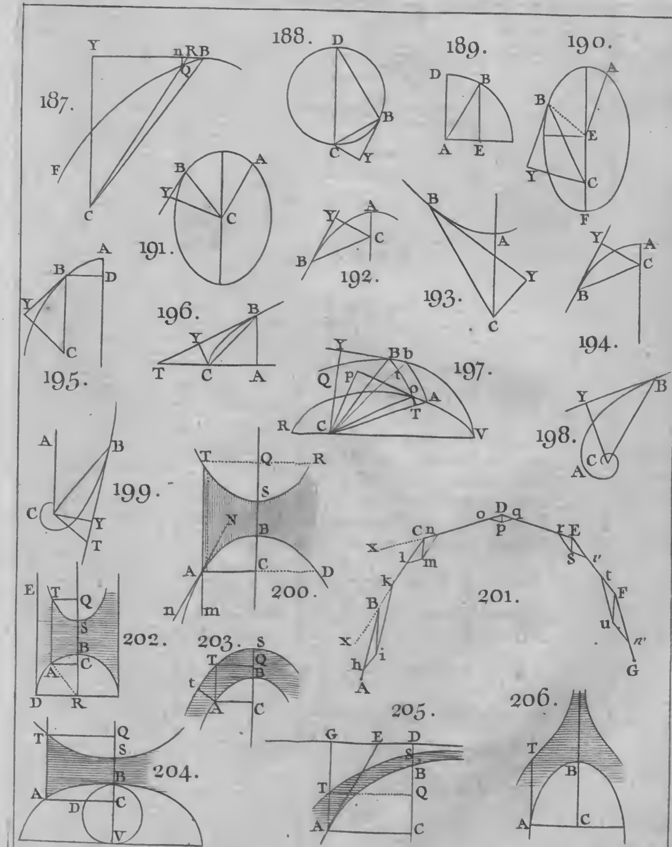
207. *The Curve BA being given, by whose Revolution about the Axis BC there is generated a concave Surface or Vault; To find the Height AT of a Wall standing on the same, and supported by that Surface, so that all the Parts may remain in Equilibrio.*

Let PBQ be an infinitely small Part of the Surface contained between the Planes PBR and QBR, draw the Ordinates DC, AC; and take Dd, Aa infinitely small equal Parts of the Curve. Let $BC=x$, $CA=y$, $BA=z$: By the Reasoning in the last Problem, the Weight insisting on the small Part of the Surface $ADda$ will be as $\frac{1}{R\dot{y}^2}$, when the Particle of the Curve is given. But this incumbent Weight is as $AD \times AT$, but $AD \propto y$, therefore the Weight is as $AT \times y\dot{y}$ and this $\propto \frac{1}{R\dot{y}^2}$, whence $AT \propto \frac{1}{Ry\dot{y}^3}$.

Or in general $AT \propto \frac{\dot{z}^3}{Ry\dot{y}^3}$. Whence

Putting $BC=x$, $CA=y$, $BA=z$. *Take* $AT \propto \frac{-\dot{x}\ddot{y}}{y\dot{y}^3}$ *if* $\dot{x}$ *be given, or* $AT \propto \frac{\ddot{x}}{y\dot{y}^2}$ *if* $\dot{y}$ *is given:*

And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And the Nature of the Curve ST will be known by expunging x. FIG.

Example 1.

Let BA be a cubic Parabola, $r^2x=y^3$; then $r^2\dot{x}=3y^2\dot{y}$, and (if $\dot{y}$ be given) $r^2\ddot{x}=6y\dot{y}^2$, whence $\frac{\ddot{x}}{y\dot{y}}=\frac{6}{rr}$; therefore AT is as $\frac{6}{rr}$ a given Quantity: Consequently the Curve ST is the same Parabola with BA, but placed in a higher Position. 203.

Ex. 2.

Let AB be a biquadratic Parabola $r^3x=y^4$, and $\dot{y}$ given. Then $r^3\dot{x}=4y^3\dot{y}$, and $r^3\ddot{x}=12y^2\dot{y}^2$, wherefore $\frac{\ddot{x}}{y\dot{y}^2}=\frac{12y}{r^3}$, whence $AT \propto y$ or AC. 208.

Ex. 3.

Let BA be a Circle, $y=\sqrt{2rx-xx}$, $\dot{x}$ given; then $\dot{y}=\frac{r\dot{x}-x\dot{x}}{\sqrt{2rx-xx}}$, $\ddot{y}=\frac{-rr\dot{x}^2}{\overline{2rx-xx}|^{\frac{3}{2}}}$; whence $\frac{-\dot{x}\ddot{y}}{y\dot{y}^3}=\frac{rr}{y\times\overline{r-x}|^3}$; therefore $AT \propto \frac{1}{y\times\overline{r-x}|^3}$ that is as $\frac{1}{AC\times CR^3}$. Whence the Perpendiculars RS, DE are Assymptotes to the Curve ST. 209.

PROB.

PROB. XXII.

FIG. *To find the Resistance of a plane Figure or Solid moving in a Fluid, in the Direction of it's Axis.*

210. Let ABQ be any plane Figure or Solid whose Axis is AQ; draw CBf parallel to the Axis AQ, and gD and Ordinate BE Perpendiculars thereto; BD a Tangent at B, and Df perpendicular to it. Call AE, x; EB, y; AB, z; Bn, $\dot{z}$; Br, $\dot{x}$; rn, $\dot{y}$.

Let fB represent the Force or Resistance of a Particle of the Fluid, striking against C with a given Velocity, then will fD be the Force against the Curve Line or Surface at B in Direction fD; and fg will be the Force or Resistance against the Curve in Direction BC, which alone is the Resistance that hinders or opposes it's progressive Motion in Direction of the Axis. But by similar Triangles $fB : fg :: fB^2 : fD^2 :: DB^2 : Dg^2 :: \dot{z}^2 : \dot{y}^2$, and $fg = \frac{\dot{y}^2}{\dot{z}^2} \times fB$. Therefore the Force of a Particle against C and B are as fB and $\frac{\dot{y}^2}{\dot{z}^2} \times fB$, that is as 1 to $\frac{\dot{y}^2}{\dot{z}^2}$. Now the Quantity of Fluid striking against Bn in the Curve is as $\dot{y}$, and against Bn in the Solid (generated by AB revolving round it's Axis) as $y\dot{y}$: Therefore the Force against the Base : Force against the Curve :: $\dot{y}$ to $\frac{\dot{y}^3}{\dot{z}^2}$, or as $\dot{y}$ to $\frac{\dot{y}^3}{\dot{x}^2+\dot{y}^2}$, or as $\dot{y}$ to $\frac{\dot{y}}{1+\frac{\dot{x}^2}{\dot{y}^2}}$, that is as $\dot{y}$ to $\frac{\dot{y}}{1+\frac{\dot{x}^2}{\dot{y}^2}}$: And Force against the Base : Force against the Solid, is as $y\dot{y}$ to $\frac{y\dot{y}}{1+\frac{\dot{x}^2}{\dot{y}^2}}$.

Note,

Note, by the Resistance of a plane Figure moving in a Fluid is meant the Resistance of a prismatic Solid of any given Depth, and whose Base is that Figure: And it is supposed to move in a Direction parallel to that Base. FIG.

Hence to find the Resistance; by the Equation of the Curve, exterminate $\dot{x}^2$ *out of the Quantity* $\dfrac{\dot{y}}{1+\dfrac{\dot{x}^2}{\dot{y}^2}}$ *for the Curve; and find the Fluent F; or out of the Quantity* $\dfrac{y\dot{y}}{1+\dfrac{\dot{x}^2}{\dot{y}^2}}$ *for the Solid, and find it's Fluent G. Then will the Resistance against the Base : to the Resistance against the Curve : : y : F. Or the Resistance against the Base : to the Resistance against the Solid : : as* y^2 *to* $2G$.

Ex. 1.

Let there be a Triangle (or prismatic Solid) ARS moving in Direction QA. Let $AQ=b$, $QR=c$, $AR=d$, 211.
$AE=x$, $EB=y$; and by similar Triangles $x=\dfrac{by}{c}$, whence $\dot{x}^2=\dfrac{bb\dot{y}^2}{cc}$, therefore $\dfrac{\dot{y}}{1+\dfrac{\dot{x}^2}{\dot{y}^2}}=\dfrac{\dot{y}}{1+\dfrac{bb}{cc}}$ $=\dfrac{cc\dot{y}}{cc+bb}=\dfrac{cc\dot{y}}{dd}$, whose Fluent $F=\dfrac{ccy}{dd}$. Whence the Resistance of the Base to that of the Side, is as y to $\dfrac{ccy}{dd}$, that is as dd to cc.

Ex. 2.

Let ARS be a Cone, whose Axis is AQ; then $x=\dfrac{by}{c}$, $\dot{x}^2=\dfrac{bb\dot{y}^2}{cc}$, and $\dfrac{y\dot{y}}{1+\dfrac{\dot{x}^2}{\dot{y}^2}}=\dfrac{y\dot{y}}{1+\dfrac{bb}{cc}}=\dfrac{ccy\dot{y}}{cc+bb}$ 211.

FIG. $\frac{cc y\dot{y}}{cc+bb} = \frac{cc y\dot{y}}{dd}$, whofe Fluent $G = \frac{ccyy}{2dd}$, and $2G = \frac{ccyy}{dd}$. Therefore the Refiftance of the Bafe to that of the Side is as yy to $\frac{ccyy}{dd}$, or as dd to cc.

Ex. 3.

210. *Let ABR be a Circle*, Radius $AQ = r$, $AE = x$, $EB = y$. Then $\dot{x} = \frac{y\dot{y}}{r-x} = \frac{y\dot{y}}{\sqrt{rr-yy}}$, and $\frac{\dot{x}^2}{\dot{y}^2} = \frac{yy}{rr-yy}$; therefore $\frac{\dot{y}}{1+\frac{\dot{x}^2}{\dot{y}^2}} = \frac{\dot{y}}{1+\frac{yy}{rr-yy}} = \frac{rr-yy}{rr}\dot{y}$, whofe Fluent $F = y - \frac{y^3}{3rr}$. Whence the Refiftance againft the Bafe, to the Refiftance againft the Circle (or cylindric Surface) is as y to $y - \frac{y^3}{3rr}$, that is as $3rr$ to $3rr-yy$: which when $y = r$, is as 3 to 2.

Ex. 4.

210. *Let ABQ be an Hemifphere*, then $\frac{y\dot{y}}{1+\frac{\dot{x}^2}{\dot{y}^2}} = \frac{rr-yy}{rr} \times y\dot{y}$, whofe Fluent $G = \frac{y^2}{2} - \frac{y^4}{4rr}$; therefore the Refiftance againft the Bafe, to the Refiftance againft the convex Surface, is as y^2 to $y^2 - \frac{y^4}{2rr}$, or as $2rr$ to $2rr-yy$; which when $y = r$, is as 2 to 1.

Ex. 5.

210. *Let ABRQ be a Spheroid*, Q the Center, $QA = a$, Latus Rectum $= 2r$, $AE = x$, $BE = y$, $QE = u = a-x$; then

then $uu = aa - \frac{a}{r}yy$, and $\dot{x} = -\dot{u} = \frac{ay\dot{y}}{r\sqrt{aa-\frac{a}{r}yy}}$; FIG.

therefore $\frac{y\dot{y}}{1+\frac{\dot{x}^2}{\dot{y}^2}} = \frac{rra-ryy}{rra+\overline{a-r}\times yy}y\dot{y}$; whose

Fluent, by Form the 4th and 11th, is $G = \frac{-ryy}{2a-2r} + \frac{\frac{1}{2}rraa\times 2.30258}{\overline{a-r}^2}\times$ Log: $\frac{rra+\overline{a-r}\times yy}{rra}$: Whence the Resistance of the Base to the Resistance of the convex Surface, is as yy to $2G$; and when $y = QR$, it will be as $a-r$ to $\frac{ar}{a-r}\times$ Log: $\frac{a}{r}$: $-r$.

Ex. 6.

Let ABE be an Hyperboloid; denoting the Quantities as in the last Example, $uu = aa + \frac{a}{r}yy$, and 210.

$\frac{y\dot{y}}{1+\frac{\dot{x}^2}{\dot{y}^2}} = \frac{rra+fryy}{rra+\overline{r+a}\times yy}y\dot{y}$, whose Fluent G

$= \frac{1}{2}\times\frac{ryy}{r+a} + \frac{1}{2}\times\frac{rraa}{\overline{r+a}^2}\times 2.30258\,\text{Log.}\frac{rra+\overline{a+r}xyy}{rra}$.

Therefore Resistance of the Base : Resistance of the convex Surface : : yy to $2G$.

Ex. 7.

Let ABE be a Paraboloid, $AE = x$, $BE = y$, 210.
$rx = yy$, then $\dot{x}^2 = \frac{y^2\dot{y}^2}{rr}$; and $\frac{y\dot{y}}{1+\frac{\dot{x}^2}{\dot{y}^2}} =$

$\frac{rry\dot{y}}{rr+yy}$, whose Fluent $G = 2.3025rr\text{Log}:\sqrt{rr+yy}$:

 And

FIG. And by Correction $G = 2.302585rr \times \text{Log:} \frac{\sqrt{rr+yy}}{r}$:

And the Resistance of the Base, to the Resistance of the Surface of the Solid, is as yy to $2.302585rr \times \text{Log:} \frac{rr+yy}{rr}$:

Ex. 8.

210. *Suppose AB to be a cubic Paraboloid* $r^2x = y^3$, then $\dot{x}^2 = \frac{9y^4\dot{y}^2}{r^4}$, whence $\frac{y\dot{y}}{1+\frac{\dot{x}^2}{\dot{y}^2}} = \frac{r^4y\dot{y}}{r^4+9y^4}$, whose Fluent G (by Form the 5th) $= \frac{,01745rr}{6}$ $\times$ Degrees in the Arch whose Tangent is $\frac{3yy}{rr}$; and the Resistance of the Base : Resistance of the Surface : : as yy to $2G$.

Ex. 9.

212. *Let ABDQ be the Solid generated by the Cycloid ABD revolving round AQ.* $DQ = a$, $AE = x$, $EB = y$, $AB = z$; by the Nature of the Curve $y = z - \frac{zz}{4a}$, and $\dot{y} = \dot{z} - \frac{z\dot{z}}{2a}$; then $\frac{y\dot{y}}{1+\frac{\dot{x}^2}{\dot{y}^2}}$

$$= \frac{y\dot{y} \times \dot{y}^2}{\dot{z}^2} = \frac{y\dot{y} \times \overline{2a-z}^2}{4aa} = \frac{\overline{a-y} \times y\dot{y}}{a},$$

whose Fluent $G = \frac{yy}{2} - \frac{y^3}{3a}$: Whence the Resistance of the Base, to the Resistance of the Surface : : as yy, to $yy - \frac{2y^3}{3a}$, or as 1 to $1 - \frac{2y}{3a}$, and in the whole Solid it is as 3 to 1.

The

The Problems delivered in this Section are exceeding general, each of them comprehending an infinite Number of particular Caſes, and are ſufficient here to ſhew the Method of inveſtigating general Problems by the Method of Fluxions. I ſhall now proceed to exemplify the ſame Doctrine in the Reſolution of a few particular Problems belonging to Phyſics or Natural Philoſophy; and the rather becauſe theſe Sort of Problems have not been ſo common among the Writers of Fluxions.

SECT. III.

The Solution of Physical Problems, or such as occur in the Phænomena of Nature.

PROB. I.

To find the Curve which a flexible Line QAR is put into by the Wind or any Fluid moving against it.

FIG. 213. LET AE be the Axis, and BE, Cf Ordinates infinitely near, Bd parallel to AE. Call AE, x; EB, y; AB, z; Bd, $\dot{x}$; Cd, $\dot{y}$; BC, $\dot{z}$; and let $\dot{z}$ be given. The Force of the Fluid acting perpendicularly against the Particle of the Curve $\dot{z}$, is as the Quantity of the Fluid acting on it, and the Sine of Incidence, that is as $\dot{y} \times \frac{\dot{y}}{\dot{z}}$, or $\frac{\dot{y}^2}{\dot{z}}$ that is as $\frac{\dot{y}^2}{\dot{z}}$; And (by Ex. 16, Prop. XIII,) that Force will be as the Curvature in B, or reciprocally as the Radius of Curvature, that is (by Prob. V. Sect. II.) as $\frac{\dot{z}\dot{x}}{-\ddot{y}}$; therefore $\frac{\dot{y}^2}{\dot{z}} \propto \frac{-\ddot{y}}{\dot{z}\dot{x}}$, or $\frac{\dot{y}^2}{\dot{z}} = \frac{-a\ddot{y}}{\dot{x}}$; therefore $\frac{\dot{x}}{\dot{z}} = \frac{-a\ddot{y}}{\dot{y}^2}$, and the Fluent is $\frac{x}{\dot{z}} = \frac{a}{\dot{y}}$;

$\frac{a}{\dot{y}}$; but in A, $\dot{y}=\dot{z}$, and $x=0$, therefore the corrected Fluent is $\frac{x}{\dot{z}} = \frac{a}{\dot{y}} - \frac{a}{\dot{z}}$, whence $x\dot{y} = a\dot{z} - a\dot{y}$, therefore $\overline{a+x} \times \dot{y} = a\dot{z}$, and $\overline{a+x}^2 \times \overline{\dot{z}^2 - \dot{x}^2} = a^2\dot{z}^2$, or $\overline{2ax+xx} \times \dot{z}^2 = \overline{a+x}^2 \times \dot{x}^2$, and $\dot{z} = \frac{\overline{a+x} \times \dot{x}}{\sqrt{2ax+xx}}$; and the Fluent $z = \sqrt{2ax+xx}$; an Equation to the Catenary. FIG.

PROB. II.

To find the Motion of a musical String, vibrating at very small Distances.

1. Let AB be the String, and let it be drawn to C, and there let go; now since the Force to move the Point C, by Mechanics, is as the Sine of the Angle ACn, or as that Angle it self when it is very small; therefore the Point C alone will first begin to move, and presently by the Flexure of the String in d and e these Points will also begin to move, and then the next Points to these, and so forward. Now by Reason of the great Flexure in C, that Point will at first be very swiftly moved; and the Curvature in d and e being thereby increased, these Points are continually accelerated; and the Curvature in C being diminished, it's Motion will be less accelerated. And universally these Points that are too slow being more accelerated, and those too swift being less accelerated, it will come to pass that, the Forces being at length rightly adjusted all the Points of the String will acquire such Motions, as to be carried to the Axis together; and will continue to go and return together *ad infinitum*. 214.

2. Now

FIG. 214. 2. Now that this may be regularly performed, the String must always have the Form of the Curve $AFXB$, whose Nature is such, that the Angle of Contact, or the Curvature in any Point F, will be as the Ordinate FE; for then the Force at F being as the Curvature that is as FE, the Velocity generated will also be as FE the Space to be described and the consequent Accelerations and Velocities, and the Parts of the Ordinates described, and those to be described, will be as the wholes; and consequently any correspondent Parts of the Ordinates, and therefore the whole Ordinates will be described in equal Times.

3. To find the Radius of Curvature; Let AB or $2AZ=a$, $ZX=b$, $AE=x$, $EF=y$, $AF=z$, $e=$ Radius of Curvature in the middle Point X. Let $\dot{z}$ be given; and by Prob. V. Sect. II. the Radius of Curvature in F $=\frac{\dot{z}\dot{y}}{\ddot{x}}$; therefore by the Nature of the Curve $y:b::e:\frac{be}{y}=\frac{\dot{z}\dot{y}}{\ddot{x}}$, and $eb\ddot{x}=\dot{z}y\dot{y}$, and the Fluent is $eb\dot{x}=\frac{yy\dot{z}}{2}$, but in X, $\dot{x}=\dot{z}$, and $y=b$: Therefore the Fluent corrected is $eb\dot{x}-eb\dot{z}=\frac{yy-bb}{2}\dot{z}$, and $eb\dot{x}=eb\dot{z}+\frac{yy-bb}{2}\dot{z}=\frac{2eb+yy-bb}{2}\sqrt{\dot{x}^2+\dot{y}^2}$, and by Reduction, $\dot{x}=\frac{\overline{2eb+yy-bb}\times\dot{y}}{\sqrt{4eb^3-4eby^2-y^4-b^4+2bby}}$ $=$ (because e is vastly greater than b or y) $\frac{2eb\dot{y}}{\sqrt{4eb^3-4ebyy}}$ $=\frac{\dot{y}\sqrt{eb}}{\sqrt{bb-yy}}$. Whence (by Form the 10th) the Fluent $x=\sqrt{eb}\times$ Arch whose Sine is $\frac{y}{b}$, Radius $=1$; and when $y=b$, then $x=\frac{1}{2}a$, and then $x=\sqrt{eb}\times\frac{3.1416}{2}$

$\frac{3.1416}{2} = \frac{c}{2}\sqrt{eb}$, putting $c = 3.1416$. Therefore FIG. 214.
$\frac{1}{2}a = \frac{c}{2}\sqrt{eb}$, whence $e = \frac{aa}{ccb}$, the Radius of Curvature in X. Therefore the Radius of Curvature in $F = \frac{aa}{ccy}$.

4. To find the Motion of any Particle of the String as suppose of X the middle Point. Let $p =$ Tension of the String, or the Force that extends it; $n =$ Weight of the String; $IZ = x$, $v =$ Velocity in I, $t =$ Time of describing XI; $\acute{x}$, $\acute{v}$, $\acute{t}$, the Moments of x, v, and t. The Radius of Curvature in I is $= \frac{aa}{ccx}$. By Ex. 16. Prop. XIII. the Force wherewith any Particle of the Curve at I is urged; is to the Tension of the String (p) : : as that Particle ($\acute{z}$): to the Radius of Curvature in I ($\frac{aa}{ccx}$); therefore the Force acting at $I = \frac{pccx\acute{z}}{aa}$. Now by Mechanics or the Laws of Motion, the Velocity α $\frac{\text{Moment of Space}}{\text{Moment of Time}}$ universally, and likewise the Moment of Velocity α $\frac{\text{Force} \times \text{Moment of Time}}{\text{Matter}}$; therefore the Velocity $\times$ Moment of Velocity α $\frac{\text{Moment of Space} \times \text{Force}}{\text{Weight}}$ (since the Weight is as the Matter); and this is an universal Proportion for these Quantities.

Now it is known that any heavy Body falling through $16\frac{1}{12}$ or f Feet gains a Velocity of $2f$ in 1 Second; therefore $\acute{x} =$ Velocity generated by that Body in falling through $\acute{x}$ with that Velocity $2f$, because the Velocities generated are as the Times, or as the Spaces ($2f$ and $\acute{x}$) uniformly described with the given Velocity $2f$. Therefore, in this Case of falling Bodies,

FIG. 214. Bodies, we have the Velocity × Moment of Velocity $= 2f\dot{x}$: and likewise $\frac{\text{Force} \times \text{Moment of Space}}{\text{Weight}} = \frac{p\dot{x}}{p} = \dot{x}$. Lastly, in the Case of the vibrating String $v\dot{v} =$ Velocity × Moment of Velocity. And, because the String is homogeneous $a : n :: \dot{z} : \frac{n\dot{z}}{a} =$ Weight of $\dot{z}$; and $-\dot{x} =$ Moment of Space. Therefore (by the Rules Prop. XIII.) we get this Analogy (from the general Proportion before laid down) $2f\dot{x} : \dot{x} :: v\dot{v} : \frac{-\dot{x} \times pccx\dot{z}}{aa \times \frac{n\dot{z}}{a}}$. Whence $v\dot{v} = \frac{-2fpccx\dot{x}}{na}$, or $v\dot{v} = \frac{-2fpccx\dot{x}}{na}$; whence the Fluent is $vv = \frac{-2fpccx^2}{na}$: but in X, $v=0$, $x=b$; therefore the Fluent corrected (by Prop. XII.) is $vv = \frac{2fpcc}{na} \times \overline{bb-xx}$. And in Z, where $x=0$, $v = bc\sqrt{\frac{2fp}{na}}$, the Feet described in a Second.

5. Lastly for the Time. Since the Moment of the Time $\propto \frac{\text{Moment of Space}}{\text{Velocity}}$, universally. And in a falling Body, $2f$ (Space) : 1 Second (Time) :: $\dot{x}$ (Space) : $\frac{\dot{x}}{2f} =$ Time of describing $\dot{x}$ by the falling Body. And likewise $\frac{\dot{x}}{2f} = \frac{\text{Space}}{\text{Velocity}}$; therefore from the general Analogy, we get $\dot{t} = \frac{-\dot{x}}{v}$, or $\dot{t} = \frac{-\dot{x}}{c\sqrt{\frac{2fp}{na}} \times \sqrt{bb-xx}}$, then the Fluent $t = \frac{-A}{c\sqrt{\frac{2fp}{na}}}$ (putting $A =$ Arch whose Sine is $\frac{x}{b}$ and Radius 1).

And

And by Correction (for in X, $t=0$, $x=b$;) $t=\frac{B}{c\sqrt{\frac{2fp}{na}}}$ (putting $B=$ Arch whose Cosine is $\frac{x}{b}$); FIG. 214.

and when $x=0$, the whole Time $t=\frac{1}{2}\sqrt{\frac{na}{2pf}}$; and $2t$ or the Time of one entire Vibration $=\sqrt{\frac{na}{2pf}}$, in Seconds.

Cor. 1. Hence all the Vibrations great and small are performed in equal Times; for they are all expressed by $\sqrt{\frac{na}{2pf}}$, in which b is not concerned.

Cor. 2. The Number of Vibrations performed in one second is $\sqrt{\frac{2fp}{na}}$.

Cor. 3. Hence the Square of the Time of Vibration of any musical String, is as it's Length and Weight directly, and it's Tension reciprocally.

SCHOLIUM.

I took a virginal String $29\frac{56}{100}$ Inches long, and weighing $8\frac{6}{10}$ Grains; and fastning it to the Virginal, I stretched it with $8\frac{1}{2}$ Pound Weight Avoirdupoise; and causing it to vibrate, I found it to be Unison with the Note Ela in the Base (the Note below the Cliff): By this Problem it appears, that the String made 300 Vibrations in a Second of Time. This Experiment I made very accurately. However, by Reason of the Resistance of the Air, and the larger Vibrations that the String makes, it is probable that the Time is a little prolonged; and that the Number of Vibrations in a Second may be something less than is assigned by this Problem.

PROB. III.

FIG. 215. *To find the Velocity of a Projectile at A moving in any given Curve QAO about the Center of Force S.*

Let the Distance $SA=D$, SB the Perpendicular on the Tangent at $A=P$, Radius of Curvature $CA=R$, $c=$ Velocity of the Body at A, $e=$ Velocity of a Body in a Circle at the same Distance SA, and acted on with the same Force; take the infinitely small Arch Aa, and draw am, an, parallel to SA, CA.

Then by similar Triangles, $P : D :: an : am ::$ centripetal Force tending to C : to centripetal Force tending to S :: versed Sine of the Arch Aa : versed Sine of the Arch (described in the same Time) whose Radius is $SA :: \frac{cc}{R} : \frac{ee}{D}$. Therefore $PR : DD :: cc : ee$.

Or thus, by Prob. V. Sect. II. $R = \frac{D\dot{D}}{\dot{P}}$; therefore $cc : ee :: P \times \frac{D\dot{D}}{\dot{P}} : DD$; that is $cc : ee :: P\dot{D} : D\dot{P}$.

COR. 1. In the Ellipsis and Hyperbola, the Square of the Velocity of a Projectile moving round the Focus : is to the Square of the Velocity of a Body moving in a Circle at the same Distance :: as the Projectiles Distance from the other Focus : is to the Semi-transverse.

For let $2r=$ Transverse, $2b=$ Conjugate; then (by Ex. 3d and 5th, Prob. XIX. Sect. II.) $P = \frac{bD}{\sqrt{2rD \mp DD}}$,

$\frac{bD}{\sqrt{2rD \mp DD}}$, and $\dot{P} = \frac{brD\dot{D}}{\overline{2rD \mp DD}|^{\frac{3}{2}}}$. Therefore FIG. 215.

$cc : ee :: \frac{bD\dot{D}}{\sqrt{2rD \mp DD}} : \frac{brD^2\dot{D}}{\overline{2rD \mp DD}|^{\frac{3}{2}}} :: 2r \mp D : r$.

Cor. 2. The Velocity of a Body revolving in an Ellipsis round the Center, is to the Velocity of a Body in a Circle at the same Distance; as the Conjugate to that Line of Distance, to the Distance itself.

For (by Ex. 4. Prob. XIX. Sect. II.) $P = \frac{br}{\sqrt{rr + bb - DD}}$, and $\dot{P} = \frac{brD\dot{D}}{\overline{rr + bb - DD}|^{\frac{3}{2}}}$, whence $cc : ee :: (\frac{br\dot{D}}{\sqrt{rr + bb - DD}} : \frac{brD^2\dot{D}}{\overline{rr + bb - DD}|^{\frac{3}{2}}} ::)$ $rr + bb - DD : DD$. And $c : e ::$ Conjugate of D : to D.

Cor. 3. The Velocity of a Body moving in a Parabola about the Focus, is to the Velocity in a Circle at the same Distance :: as $\sqrt{2}$ to 1.

For let $r =$ Latus Rectum, then by the Nature of the Parabola $P = \frac{\sqrt{rD}}{2}$, and $\dot{P} = \frac{\dot{D}}{4}\sqrt{\frac{r}{D}}$. Whence $cc : ee :: (\frac{\dot{D}\sqrt{rD}}{2} : \frac{D\dot{D}}{4}\sqrt{\frac{r}{D}} ::)\ 2 : 1$.

PROB. IV.

FIG. 216. *To find the Velocity of a descending Body in any Place P, let fall from the given Point D towards the Earth; being acted upon by a Force which is as any Power of it's Distance CP from the Center.*

Let the Earth's Radius $CA=r$, $CD=a$, $CP=x$, $DP=a-x$, whose Fluxion is $-\dot{x}$, $t=$ Time of descending through DP, $v=$ Velocity acquired by that Descent. F the Force at P, which let be as x^n.

By Mechanics, when the Body is given, it is universally $\dot{v} \propto F\dot{t}$, and $\dot{t} \propto \frac{-\dot{x}}{v}$, for any Velocities, Times and Spaces; therefore $\dot{v} \propto \frac{-F\dot{x}}{v} \propto \frac{-x^n\dot{x}}{v}$; and $v\dot{v} \propto -x^n\dot{x}$, and the Fluent $v^2 \propto \frac{-x^{n+1}}{n+1}$. But in D, $v=0$, $x=a$, therefore the Fluent corrected is $v^2 \propto \frac{a^{n+1}-x^{n+1}}{n+1}$. But if $n=-1$, then $vv \propto \text{Log.}\frac{a}{x}$.

Now we must find the Value of v at A the Earth's Surface for some determinate Values of a and x, in order to turn the general Proportion into an Equation. Thus, it is known by Experiments that a heavy Body descending through a Space s or $16\frac{1}{12}$ Feet will acquire a Velocity of $2s$ or $32\frac{1}{6}$ Feet in a Second of Time. Therefore writing $2s$ for v, r for x, $r+s$ for a, we shall get this Analogy, $\frac{\overline{r+s}^{n+1}-r^{n+1}}{n+1}$:

FIG.

$4ss :: \frac{a^{n+1} - x^{n+1}}{n+1} : vv = 4ss \times \frac{a^{n+1} - x^{n+1}}{\overline{r+s}^{n+1} - r^{n+1}}$
$=$ (because $\overline{r+s}^{n+1} - r^{n+1} = \overline{n+1} \times r^n s$ nearly) $4ss \times$
$\frac{a^{n+1} - x^{n+1}}{\overline{n+1} \times r^n s}$; and $v = 2s\sqrt{\frac{a^{n+1} - x^{n+1}}{\overline{n+1} \times r^n s}}$, the Feet described in a Second.

Cor. 1. If $n = -2$, then $v = 2r\sqrt{\frac{s \times \overline{a-x}}{ax}}$, therefore the Velocity of a Body falling from an infinite Distance to the Surface of the Earth will be $2\sqrt{rs}$.

Cor. 2. If $n = 0$, then $v = 2\sqrt{s \times \overline{a-x}}$.

Cor. 3. If $n = 1$, then $v = \sqrt{\frac{2s}{r} \times \overline{aa - xx}}$. Therefore a Body falling from the Surface of the Earth to the Center acquires the Velocity $\sqrt{2rs}$.

PROB. V.

To find the Time wherein a falling Body will descend through any Space towards the Earth, being acted upon by a Force which is as any Power of it's Distance from the Earth's Center. 216.

The same Things supposed as in the last Problem, we shall have, the Moment of Time $\propto \frac{\text{Mom. Space}}{\text{Velocity}}$, universally. Since a descending Body at the Earth's Surface acquires the Velocity $2s$ in the Time p or 1 Second, therefore by the Laws of uniform Motion, $2s : p :: \dot{x} : \frac{p\dot{x}}{2s} =$ Moment of Time, wherein $\dot{x}$ is

is described with Velocity $2s$. Hence from the universal Proportion, $\frac{p\dot{x}}{2s} : \frac{\dot{x}}{2s} :: t' : \frac{-\dot{x}}{v}$; therefore $t' = \frac{-p\dot{x}}{v} = \frac{-\dot{x}}{v}$ (because $p = 1$). And $\dot{t} = \frac{'-\dot{x}}{v} =$ (by the last Problem) $\frac{-\dot{x}}{2s\sqrt{\frac{a^{n+1} - x^{n+1}}{\overline{n+1} \times r^n s}}}$;

whence $t =$ Fluent of $\frac{-\dot{x}}{2s\sqrt{\frac{a^{n+1} - x^{n+1}}{\overline{n+1} \times sr^n}}}$, expressed in Seconds.

COR. 1. If $n = -2$, $t =$ Fluent of $-\frac{1}{2r}\sqrt{\frac{a}{s}} \times \frac{x^{\frac{1}{2}}\dot{x}}{\sqrt{a-x}}$ $=$ (by Form 10 and 11) $\frac{1}{2r}\sqrt{\frac{a}{s} \times \overline{ax-xx}} - \frac{a}{2r}\sqrt{\frac{a}{s}}$ $\times$,017453 $\times$ Degrees in the Arch whose Sine is $\sqrt{\frac{x}{a}}$, and Radius 1. And when duly corrected, the Time $t = \sqrt{\frac{aax - axx}{4rrs}} + \frac{,017453a}{2r}\sqrt{\frac{a}{s}} \times$ Degrees in the Arch whose Cosine is $\sqrt{\frac{x}{a}}$, and Radius 1. And the Time of descending to the Center is $\frac{3.1416a}{4r}\sqrt{\frac{a}{s}}$.

COR. 2. If $n = 0$, then $\dot{t} = \frac{-\dot{x}}{\sqrt{4s \times \overline{a-x}}}$, and $t = \sqrt{\frac{a-x}{s}}$.

COR. 3. If $n = 1$, then $\dot{t} = \sqrt{\frac{r}{2s}} \times \frac{-\dot{x}}{\sqrt{aa-xx}}$, and (by Form 10) $t = -\sqrt{\frac{r}{s}} \times$,017453 Degrees of the Arch

Arch whose Sine is $\frac{x}{a}$, and Radius 1. And being FIG.
duly corrected, $t = ,017453\sqrt{\frac{r}{2s}} \times$ Number of Degrees in the Arch whose Cosine is $\frac{x}{a}$, and Radius 1. Hence the Time of descending to the Center will be $\frac{3.1416}{2}\sqrt{\frac{r}{2s}}$: And therefore all the Times of Descent from any Altitudes whatsoever will be equal.

SCHOLIUM.

If t be given to find x; Find the Fluent of $\dot{t} = -\frac{1}{2}r\sqrt{\frac{a}{s}} \times \frac{x^{\frac{1}{2}}\dot{x}}{\sqrt{a-x}}$, or $\sqrt{\frac{2r}{s}} \times \frac{-\dot{x}}{\sqrt{aa-xx}}$, by infinite Series and revert the Series. And if either t or v be given the other may be found by first finding x. And hence a Body being projected upwards with any Velocity, it's Height may be found, and the Time of it's Ascent.

PROB. VI.

The Velocity and Direction of a Projectile, and the Law of centripetal Force being given; to find the Velocities, Times and Angles of Revolution.

Let C be the Center of Force, and let the Body be projected from V in Direction VA with Velocity b describing the Space b in the Time g; and let p be the Velocity and q the Space which the Force at V will generate in the same Time g. To the Center C describe the Circle VXH, draw the Radii CX, CY infinitely 217.

FIG. 217. finitely near, cutting the Trajectory in I and K, and describe the Arch Kr. Call CV, a; CI, x; VI, u; VX, z; Ir, $\dot{x}$; Kr, $\dot{y}$; YX, $\dot{z}$; IK, $\dot{u}$: And let $t =$ Time of describing VI, and t' of describing IK, $s =$ Sine, $c =$ Cosine of the Angle CVA, and let the Force in any Place I be as x^n, and $v =$ Velocity in I. Then

1. By the Resolution of Forces, the Force to accelerate the Body in Direction of the Curve is $x^n \times \frac{Ir}{IK} = x^n \times \frac{-\dot{x}}{\dot{u}}$; but by the Laws of uniform Motion $t' \propto \frac{\dot{u}}{v}$, and $\dot{v} \propto \text{Force} \times t' \propto \text{Force} \times \frac{\dot{u}}{v}$, and $v\dot{v} \propto \text{Force} \times \dot{u}$, universally. Whence $v\dot{v} \propto \frac{-x^n\dot{x}}{\dot{u}} \times \dot{u} \propto -x^n\dot{x}$. Hence therefore the Moment of Velocity depends not at all on the Angle KIC, but upon $\dot{x}$ the Moment of perpendicular Descent, and is therefore the same at all Inclinations as if the Body descended perpendicularly. Now to find the Moment of Velocity in V; by the Laws of uniformly accelerated Motion, the Velocity generated is as the Time, or as the Space uniformly described with a given Velocity: And since in the Time of describing $2q$, the Velocity p is generated, therefore $2q : p :: \dot{x} : \frac{p\dot{x}}{2q} =$ Moment of Velocity generated at V whilst $\dot{x}$ is described with Velocity p. Here therefore the Value of $v\dot{v}$ is $\frac{pp\dot{x}}{2q}$, and Force $\times$ Moment of Space is $a^n\dot{x}$. Therefore from the universal Proportion $\frac{pp\dot{x}}{2q} : a^n\dot{x} :: v\dot{v} : -x^n\dot{x} :: v\dot{v} : -x^n\dot{x}$, whence $v\dot{v} = \frac{-ppx^n\dot{x}}{2qa^n}$, and the Fluent $v^2 = \frac{-ppx^{n+1}}{\overline{n+1} \times qa^n}$. But in V, $x = 0$, and $v = b$, therefore

therefore by Correction $vv = bb + \frac{ppa}{n+1.q} - \frac{ppx^{n+1}}{n+1.qa^n}$.

But here if $n = -1$, then $vv = bb + \frac{ppa}{q} \times \text{Log.}\frac{a}{x}$.

2. Again, since the Velocity is every where reciprocally as the Perpendicular let fall on the Tangent; and these Perpendiculars in V and I are sa, and $\frac{x\dot{y}}{\dot{u}}$; therefore $b : v :: \frac{x\dot{y}}{\dot{u}} : sa$, then is $\dot{y} = \frac{asb}{vx}\dot{u} = \frac{asb}{vx}\sqrt{\dot{x}^2+\dot{y}^2}$, and by Reduction $\dot{y} = \frac{asb\dot{x}}{\sqrt{v^2x^2 - a^2s^2b^2}}$.

But by similar Triangles $x : \dot{y} :: a : \dot{z} = \frac{a\dot{y}}{x}$, and $\dot{z} = \frac{a\dot{y}}{x}$, therefore $\dot{z} = \frac{aasb\dot{x}}{x\sqrt{v^2x^2 - a^2s^2b^2}}$.

3. Lastly, since $\dot{t}' \propto \frac{x\dot{y}}{2}$, the Moment of the Area, or the Time $\propto$ Area; and in V, the Area $= \frac{asb}{2}$; therefore as $g : asb :: t' : x\dot{y} :: \dot{t} : x\dot{y}$; then will $\dot{t} = \frac{gx\dot{y}}{asb}$, or $\dot{t} = \frac{gbx\dot{x}}{b\sqrt{v^2x^2 - a^2s^2b^2}}$.

Consequently substituting for vv it's equal, in the Values of $\dot{z}$ and $\dot{t}$, the Fluents will give z and t.

COR. 1. Hence the Apsides of the Trajectory are easily found; for then it will be $\dot{u} = \dot{y} = \frac{asb}{vx}\dot{u}$, or $asb = vx$; therefore $vv = \frac{a^2s^2b^2}{xx} = bb + \frac{ppa}{n+1 \times q} - \frac{ppx^{n+1}}{n+1 \times qa^n}$. Whence x will be found: And if two Roots of this Equation be found; then the correspondent Fluents z will give the Position of the Apsides.

FIG. COR. 2. Since the Sine of the Angle $CIK = \frac{\dot{y}}{\dot{u}} = \frac{asb}{vx}$; therefore if that Angle be given the Distance x may be found; or if x be given the Angle may be found.

COR. 3. If the Body be projected at right Angles to CV, then $s=1$, and, by Cor. 1, we shall have $bbxx + \frac{ppax^2}{\overline{n+1}\times q} - \frac{ppx^{n+3}}{\overline{n+1}\times qa^n} = a^2bb$; in which one Root is a; and finding x another Root in the Equation, the Fluent z may thence be had; and consequently the Motion of the Apsides.

PROB. VII.

To find the Time of a Body's descending through any Arch of a Cycloid.

218. Let AC be the Axis, BV, DF, ef Ordinates. Let $AC=a$, $VC=b$, $VF=x$, $Ff=\dot{x}$, $BD=z$, $De=\dot{z}$, $s=16\frac{1}{12}$ Feet. $t=$ Time of describing BD: And let the Body fall from B.

The Times of describing any Spaces uniformly are as the Spaces directly, and the Velocities reciprocally; but the Velocities are as the Square Roots of the Heights fallen from; and $2s$ is the Space uniformly described in 1 Second by the Velocity acquired by falling through s: Therefore $\frac{2s}{\sqrt{s}}$: 1 Second :: $\frac{\dot{z}}{\sqrt{x}} : \dot{t} = \frac{\dot{z}}{2\sqrt{sx}}$; or $\dot{t} = \frac{\dot{z}}{2\sqrt{sx}}$; but (by Ex. 8. Prob.

Prob. VIII. Sect. II.) $\dot{z} = \dot{x}\sqrt{\frac{a}{b-x}}$; whence $\dot{t} =$ FIG. 218.

$\frac{1}{2}\sqrt{\frac{a}{s}} \times \frac{x^{-\frac{1}{2}}\dot{x}}{\sqrt{b-x}}$. Whence the Fluent (by Form the 10th) is $t = \sqrt{\frac{a}{s}} \times$ Arch whose Sine is $\sqrt{\frac{x}{b}}$, and Radius 1. And when $x = b$, then $t = \frac{3.1416}{2}\sqrt{\frac{a}{s}}$. Whence if a Pendulum be made to vibrate in the Arch of the Cycloid, $2t$ or the Time of one entire Vibration will be $3.1416\sqrt{\frac{a}{s}}$, in Seconds.

Cor. 1. Hence all the Times are equal in which Bodies descending from any Points Z, B, D shall arrive at the lowest Point C: And all the Times of Vibration will be equal among themselves.

Cor. 2. It appears by Ex. 3. Prob. VII. Sect. II. that if ZP, PQ be two Cycloids, whose Cuspids are at P, and Vertices at Z and Q; then if a Pendulum PC be suspended at P, so that in oscillating it may fold about the Curves ZP, PQ; then the Point C will describe the Cycloid ZCQ: And therefore the Time of it's Vibration will be $3.1416 \times$ Time of a Body's falling through AC.

PROB. VIII.

FIG. 219. *To find the Force wherewith a Corpuscle P is attracted to the Plane of a Circle EQD, according to any Law of centripetal Force.*

Let A be the Center, and AP perpendicular to the Plane of the Circle, and let the Force of each Particle be as the n^{th} Power of the Distance.

Put $AP = a$, $AE = x$, $c = 3.1416$. And let the Body m attract the Corpuscle P, at the Distance d, with the Force f; then the Force which any Particle $\dot{x}$ attracts the Corpuscle P towards E is $\frac{f\dot{x}}{md^n} \times \overline{aa + xx}|^{\frac{n}{2}}$, and the Force of all the Particles in the Periphery EQD is $\frac{2cf\dot{x}x}{md^n} \times \overline{aa+xx}|^{\frac{n}{2}}$: And by Mechanics the Force in Direction $PA = \frac{2cfax\dot{x}}{md^n} \times \overline{aa+xx}|^{\frac{n-1}{2}}$; and the Fluxion of the Force $= \frac{2cfax\dot{x}}{md^n} \times \overline{aa+xx}|^{\frac{n-1}{2}}$; whose Fluent is $\frac{2cfa}{md^n} \times \frac{\overline{aa+xx}|^{\frac{n+1}{2}}}{n+1}$. But in A, $x=0$; therefore by Correction, the Force exerted on P by the Plane of the Circle ED is $= \frac{2cfa}{md^n} \times \frac{\overline{aa+xx}|^{\frac{n+1}{2}}}{n+1} - \frac{2cfa^{n+2}}{\overline{n+1} \times md^n}$. But if $n = -1$; then the Force $= \frac{cfad}{m} \times 2.302585$ Log. $\frac{aa+xx}{aa}$;

COR. I.

FIG.

COR. 1. Hence if $n=1$, the Force of the Circle exerted on the Particle P will be $\frac{cfaxx}{md}$, the same as if the said Circle were wholly collected into the Center A.

COR. 2. Therefore if $n=1$, a Sphere will attract any Particle P with the same Force as if the whole Sphere was contracted into the Center C. For taking the Circles ED, ed parallel, and equidistant from the Center; the Sum of the Forces will be as the Sum of the two Circles each multiplied into it's Distance, or as either Circle into half the Sum of the Distances, that is into $2PC$; or both Circles multiplied into PC; and it is the same of all equidistant Circles that compose the Globe. 220.

COR. 3. If $n=-2$, the Force $=\frac{2cfdd}{m}\times\overline{1-\frac{a}{\sqrt{aa+xx}}}$ 219.

$=\frac{2cfdd}{m}\times\overline{1-\frac{PA}{PE}}$.

COR. 4. If n be less than -1, then the Force of the whole infinite Plane will be $\frac{2cfa^{n+2}}{md^n\times\overline{-n-1}}$.

PROB. IX.

To find the Force wherewith an infinite Solid, plain on one Side Ll, attracts a Corpuscle placed at C: Supposing the Law of Attraction to be inversely as some Power of the Distance greater than 1. 221.

Draw the infinite Line CGK, perpendicular to the Plane LGl, and through the Points I, K, infinitely near

FIG. 221. near each other, draw two Planes parallel to Ll; and let $CG = a$, $CI = x$, $IK = \dot{x}$, and the Force as x^n; and by Cor. 4. Prob. VIII. the Force wherewith the Solid contained between the Planes at I, K, attracts the Corpuscle, will be $\frac{2cfx^{n+2}\dot{x}}{md^n \times \overline{-n-1}}$, and it's Fluxion $\frac{2cfx^{n+2}\dot{x}}{md^n \times \overline{-n-1}}$, whose Fluent is $\frac{2cf}{-md^n} \times \frac{x^{n+3}}{\overline{n+1} \times \overline{n+3}}$. And when duly corrected the Force will be $\frac{2cf}{\overline{n+1} \times \overline{n+3} \times md^n} \times \overline{a^{n+3} - x^{n+3}}$.

COR. 1. If n be less than -3, and the Solid infinite towards K; the Force will be $\frac{2cfa^{n+3}}{\overline{n+1} \times \overline{n+3} \times md^n}$.

COR. 2. Hence therefore the Force at different Distances from the infinite Solid (when n is less than -3) will be as a^{n+3} or CG^{n+3}.

COR. 3. Hence also (if n be less than -3), the Force of a very great Body upon a very small Particle, at any extremely small Distances, will be as the $\overline{n+3}^{th}$ Power of the Distance, nearly.

COR. 4. And if the Corpuscle be placed within the Solid at H, so as $GH = GC$, the Force will be the very same as if it were placed at C, so far without it. For, taking $IH = GH$ the Solids HG and HI destroy one another's Effects.

PROB.

PROB. X.

To find the Force wherewith a Sphere attracts a Corpuscle P, situated either without or within the Sphere; supposing the Forces of all the Particles to be reciprocally as the Squares of their Distances. FIG.

Case 1. Let P be without the Sphere. Draw the Axis PAB, and the Ordinates ED, ed infinitely near, and let S be the Center, put $PS = a$, $PD = x$, $Dd = \dot{x}$, $AS = r$, $ED = y$, $bb = aa - rr$. Then $PE = \sqrt{xx + yy} = \sqrt{xx + rr - \overline{a - x}^2} = \sqrt{rr - aa + 2ax} = \sqrt{2ax - bb}$. But by Cor. 3. Prob. VIII. the Force of the infinitely thin Solid contained between the Planes of the Circles DE, de is $= \frac{2fcdd\dot{x}}{m} \times \overline{1 - \frac{PD}{PE}}$, therefore the Fluxion of the Force is $\frac{2cfdd}{m} \times \overline{\dot{x} - \frac{x\dot{x}}{\sqrt{2ax - bb}}}$, whose Fluent is $\frac{2cfdd}{m} \times : x - \frac{bb + ax}{3aa} \sqrt{2ax - bb}$, but in A where the Force is 0, $x = a - r$; therefore by proper Correction, the Force of the Segment EAF is $= \frac{2cfdd}{m} \times$ into : $x - a + r + \frac{bb + aa - ar}{3aa} \times \overline{a - r} - \frac{bb + ax}{3aa} \times PE$: and when $x = a + r$, the Force of the whole Sphere is $= \frac{2cfdd}{m} \times \frac{2r^3}{3aa}$. 222.

Case 2.

FIG. 223. Case 2. Let P be within the Sphere. Let $PQ=b$; $PD=x$, then $PE=\sqrt{rr-aa+2ax}=\sqrt{bb+2ax}$; therefore (by Cor. 3. Pr. VIII.) the Fluxion of the Force at D is $\frac{2cfdd}{m}\times\dot{x}-\frac{x\dot{x}}{\sqrt{bb+2ax}}$; whose Fluent is $\frac{2cfdd}{m}\times : x+\frac{bb-ax}{3aa}\sqrt{bb+2ax}$: but in P, $x=0$; therefore the Force of the Zone or Section $QEDFR = x+\frac{bb-ax}{3aa}\sqrt{bb+2ax}-\frac{b^3}{3aa}$. In which writing $a+r$ for x, there comes out $\frac{2cfdd}{m}\times\frac{r^3+a^3-b^3}{3aa}$ for the Attraction of the Segment QBR. And by a like Process the Attraction of the Segment $QAR=\frac{2cfdd}{m}\times\frac{r^3-a^3-b^3}{3aa}$: whose Difference is $\frac{2cfdd}{m}\times\frac{2}{3}a$, the absolute Force of the Corpuscle P towards the Center; which is the same as the Force of a Sphere, whose Radius is SP, acting on the Corpuscle P at it's Surface.

COR. 1. Hence the Force of the Sphere upon the Particle P placed without the Sphere is the very same as if the whole Sphere was collected into the Center, and exerted the Sum of all the Forces from that Center. For $\frac{4r^3c}{3aa}$ = Sphere divided by the Square of the Distance; and $\frac{m}{dd}:f::\frac{4r^3c}{3aa}:\frac{4cr^3d^2f}{3ma^2}$ the very same Force of the Sphere before found. And hence it is also evident, that the Forces of Spheres are accurately in the reciprocal Ratio of the Squares of the Distances from their Centers.

COR. 2. The Force wherewith any Corpuscle P within a Sphere, is attracted to the Center, is accurately as it's Distance from the Center.

 PROB.

PROB. XI.

To find the Force wherewith a Spheroid attracts a Corpuscle P, lying upon it's Surface in the Axis PB. FIG.

Let $PB=2r$, Diameter $GH=2a$, $AP=x$, then 224.
$AE=\frac{a}{r}\sqrt{2rx-xx}$, and $PE=\sqrt{xx+\frac{aa}{rr}\times\overline{2rx-xx}}$
$=\frac{1}{r}\sqrt{2ra^2x+rrxx-aaxx}=\frac{1}{r}\sqrt{2raax+bbxx}$,
putting $bb=rr-aa$. By Cor. 3. Prob. VIII. the Fluxion of the Force at A is $=\frac{2cfdd\dot{x}}{m}\times\overline{1-\frac{PA}{PE}}$
$=\frac{2cfdd}{m}\times:\dot{x}-\frac{rx^{\frac{1}{2}}\dot{x}}{\sqrt{2ra^2+bbx}}$: Let $\varphi=\frac{2.30258}{\frac{1}{2}b}\times$
Log : $\frac{b\sqrt{x}+\sqrt{2raa+bbx}}{\sqrt{2raa}}$, if r be greater than a:
or $\varphi=\frac{,017453}{\frac{1}{2}b}\times$ Degrees in the Arch, whose Sine is $\frac{b}{a}\sqrt{\frac{x}{2r}}$, if r is less than a; and the Fluent is $=\frac{2cfdd}{m}\times:x+\frac{rraa}{bb}\varphi-\frac{r}{bb}\sqrt{2ra^2x+b^2x^2}$: $=$ Force wherewith the Segment PEF attracts the Particle at P.

Cor. 1. Hence the Force wherewith the whole Spheroid attracts the Particle P is $=\frac{2cddf}{m}\times$ into $2r+\frac{r^2a^2}{bb}\varphi-\frac{2r^3}{bb}$: where $\varphi=\frac{2.302585}{\sqrt{rr-aa}}\times$

 2 Log:

FIG. $2\,\mathrm{Log}: \frac{r+\sqrt{rr-aa}}{a}$: or $\phi = \frac{,017453}{\sqrt{aa-rr}} \times$ twice the Number of Degrees in the Arch whose Sine is $\frac{\sqrt{aa-rr}}{a}$, Radius $= 1$, according as r is greater or lesser than a.

COR. 2. If b be very small, the Force of the Segment $EPF = \frac{2cdd}{m} f \times$ into $x - \frac{2rx}{3a}\sqrt{\frac{x}{2r}} + \frac{bbxx}{10a^3}\sqrt{\frac{x}{2r}}$: very near. And the Force of the whole Spheroid is $= \frac{2cfdd}{m} \times$ into $2r - \frac{4rr}{3a} + \frac{2rr}{5a^3} bb$; as will appear by infinite Series, or Form the 16th.

PROB. XII.

To find the Motion of a Ray of Light passing into a refracting Medium.

225. Let there be two Mediums separated by the refracting Space $RAdD$ terminated by the Parallel Planes RA, Dd; and let the Ray, moving in the Direction GH, pass from H to I, and in it's Passage be acted upon, in Lines perpendicular to the Planes, by any Force which is equal at equal Distances from either Plane, and at all Distances as any Powers or Sums of Powers of the Distance therefrom: And let $b =$ Velocity in H, $v =$ Velocity in P, $CP = x$, $PI = z$, $PF = \dot{z}$, $t' =$ Time of describing PF. And let the Force be as $A + Bx^m + Cx^n$ +

$+ Dx^r$ &c $= \mathcal{Q}$; then will the Force in Direction PF be $\frac{\dot{x}}{\dot{z}}\mathcal{Q}$. And since the Velocity $\propto \frac{\text{Space}}{\text{Time}}$, and Moment of Velocity $\propto$ Force $\times$ Time, therefore $v\dot{v} \propto \mathcal{Q}\dot{x}$, or $v\dot{v} \propto \mathcal{Q}\dot{x}$; assume p a given Quantity, and let $v\dot{v} = p\mathcal{Q}\dot{x}$, and let F be the Fluent of $\mathcal{Q}\dot{x}$, then $v^2 = 2pF$, and by Correction $vv - bb = 2pF$, and $vv = bb + 2pF$. Hence the Ray will always have the same Velocity in the same Medium DIK, whatever be the Angle of Incidence.

Let the Motion of the Ray GH be divided into two GA, AH, one parallel the other perpendicular to the Plane RA. Then since the parallel Motion AH is not at all changed by the Actions of the Forces perpendicular to these Planes: Therefore if ID be made $= AH$, and DR perpendicular to DI, then IK will be described in the same Time as GH. Therefore drawing IE parallel to GH, the Velocity in H to the Velocity in I, is as GH or EI to IK, that is as the Sine of the Angle of Refraction to the Sine of the Angle of Incidence. And therefore the Velocity of Light in Vacuo, to it's Velocity in Air of a mean Density at the Surface of the Earth; is as $,999\frac{3}{4}$ to 1. For by Experiment the Sines of Refraction and Incidence are in that Ratio.

Cor. 1. The Sine of the Angle of Incidence at one Plane, is to the Sine of the Angle of Emergence from the other Plane, in a given Ratio. For the Velocity v at the second Plane will always be equal to the given Quantity $\sqrt{bb+2pF}$, and the Sine of Incidence to the Sine of Emergence, as this given Quantity to b.

Cor. 2. If the Ray was to fall on I in Direction KI, with the Velocity it has at I, it would return in the same Curve IPH, and so go to G, and obtain it's first Velocity. For the same Forces that did before

FIG. accelerate it's Paſſage, will now equally retard it in returning. Therefore,

226. COR. 3. If the Ray have a greater Velocity in the firſt Medium, than in the ſecond, and the Angle of Incidence GHA be continually diminiſhed, the Ray will at laſt be reflected; and the Angle of Reflection gha will be equal to the Angle of Incidence GHA.

For let the Angle GHA be ſuch, that the Ratio of it's Coſine to the Radius may be equal or greater than the Ratio of the Sine of Incidence of the firſt Medium, to the Sine of Emergence in the ſecond; and the Ray at R will be moving in a Direction parallel to the Planes; but being afterwards acted on by the ſame Forces as before, it will be turned back deſcribing the Line Rhg ſimilar and equal to RHG, and the Angle $gha = GHA$.

COR. 4. Hence if there be two ſimilar Mediums whoſe Denſities are p and q; and the Velocities after Refraction into each of them z and y. then will $zz - bb : yy - bb :: p : q$.

For ſince the Forces of Attraction are made towards Bodies, theſe Forces will be proportional to the Cauſes that produce them, and therefore will be as the Denſities of theſe Bodies, ſuppoſing the internal Form and Conſtitution of the Bodies to be in other Reſpects the ſame. The Forces therefore exerted at any equal Diſtances by theſe two Mediums will be as $p\mathcal{Q}$ and $q\mathcal{Q}$; whence will be had $zz - bb = 2pF$, and $yy - bb = 2qF$; whence $zz - bb : zz - yy :: p : q$, that is as the Denſities of the Bodies, nearly.

COR. 5. If Light paſs through ſeveral refracting Mediums, the Sum of all the Refractions will be equal to the ſingle Refraction it would have ſuffered, by paſſing immediately out of the firſt Medium into the laſt.

For

For supposing these several Mediums to be separated by parallel Planes, the Refraction, Velocity, or Motion generating in approaching any one of these Mediums, will be destroyed again in it's receding (on the other Side) from the same Medium. And therefore the Motion of the Ray can only be affected with the Force of that Medium it at last moves in.

SCHOLIUM.

Though it is not known to what precise Distance the refractive Power of any Medium reaches; yet we are sure it is contained in an exceeding small Compass. And therefore the Curve *HPI*, and consequently the Points *H*, *I* may be taken in Practice only as one Point.

There are few or no Examples among all the Phænomena of Nature that afford so clear a Proof of the prodigious Forces of the small Particles of Matter, as the Motion and Refraction of Light does. For notwithstanding the amazing Velocity of the Rays, and the extremely small Space and Time that any refracting Surface has to act in; and yet to produce such a sensible Refraction as we see it does, must evince that the Forces exerted on these small Bodies must be surprizingly great, and do really exceed all Comprehension.

PROB. XIII.

The Velocity of a Globe, moving in a right Line, and it's Density, and the Density of the resisting Medium in which it moves being given; to find the Time, Velocity, and Space described.

Here we suppose the Medium to be uniform, and that the Projectile is acted on by no Force but the Resistance of the Medium, and that to be as the Square of the Velocity.

Let W = Weight of the Globe, d = the Diameter, q = it's Density, p = Density of the Medium, s = a Space of $16\frac{1}{12}$ Feet, b = Space the Globe at first can describe in 1 Second, or the first Velocity; x = any Space described, t = Time of describing it, and v = Velocity at the End of that Time. Here I measure the Velocity by the Space described in a Second, and the Time is Seconds.

1. It is proved by Experiments that the Resistance of the Globe is to the Force by which it's Motion may be generated in the Time of describing $\frac{8}{3}$ it's Diameter, as the Density of the Fluid to the Density of the Globe nearly. The Velocity generated in a given Body is as the Force and Time conjunctly, therefore the Force is as the Velocity divided by the Times or by Spaces uniformly described in these Times, therefore $\frac{2s}{b} : w :: \frac{b}{\frac{8}{3}d} : \frac{3bbw}{16ds}$ = Force that will generate the Globe's Motion in the Time it describes $\frac{8}{3}d$;

$\frac{8}{3}d$; therefore $\frac{3bbp}{16dsq}W =$ Resistance of the Globe with Velocity b, and likewise $\frac{3pvv}{16dsq}W =$ Resistance with the Velocity v. Now 1 Second : $2s$ (Velocity of a falling Body) : : $\dot{t}$: $2s\dot{t} =$ Velocity generated in the Time $\dot{t}$ by Gravity. Now since the Moment of Velocity is, in all Cases, as the Force and Moment of Time; therefore $2s\dot{t} : w\dot{t} :: -\dot{v} : \frac{3pvv}{16dsq}W \times \dot{t}$; therefore $\dot{v} = \frac{-3pv^2\dot{t}}{8dq}$, or $v^{-2}\dot{v} = \frac{-3p\dot{t}}{8dq}$; whence the Fluent $v^{-1} = \frac{3pt}{8dq}$. But when $t = 0$, $v = b$, therefore the Fluent corrected is $\frac{1}{v} - \frac{1}{b} = \frac{3pt}{8dq}$, which reduced is $8dqv + 3pbtv = 8dqb$.

2. 1 Second : $2s$ (Space uniformly described with Velocity $2s$) : : $\dot{t}$: $2s\dot{t}$ = Moment of Space described with Velocity $2s$. And since Moment of Velocity $\propto \frac{\text{Force}}{\text{Velocity}} \times$ Moment of Space; and the Moment of Velocity in falling Bodies was found before $= 2s\dot{t}$. Therefore $2s\dot{t} : \frac{W \times 2s\dot{t}}{2s} :: -\dot{v} : \frac{3pvvW}{16dsq} \times \frac{\dot{x}}{v}$; whence $-\dot{v} = \frac{3pv\dot{x}}{8dq}$, and $\frac{-\dot{v}}{v} = -\frac{3p\dot{x}}{8dq}$. Therefore the Fluent is $\frac{3px}{8dq} = -$ 2.302 Log. v. And by Correction $\frac{3px}{8dq} =$ 2.302585 Log : $\frac{b}{v}$:

COR. 1. Hence $x = \frac{8dq}{3p} \times 2.302585$ Log: $\overline{1 + \frac{3pbt}{8dq}}$: this appears by expunging v.

COR. 2.

COR. 2. If $T = \frac{8dq}{3bp}$, then $v = \frac{Tb}{T+t}$; and $x = bT \times 2.302585 \text{ Log:} \frac{T+t}{T}$; and $\text{Log:} \frac{v}{b} = \frac{-x}{2.302585bT}$.

SCHOLIUM.

It is here laid down as a Principle that the Reſiſtance of a Globe moving in a reſiſting Medium, is to the Force by which it's Motion may be generated in the Time of deſcribing $\frac{2}{3}$ it's Diameter; as the Denſity of the Medium to the Denſity of the Globe: Yet I have found by ſome Experiments that in ſwift Motions, the Reſiſtance has been greater ſometimes by a third or fourth Part. Theſe Experiments I tried in a River, with a Globe of equal Denſity with the Water, by faſtning a Thread to the Globe and to an Index incloſed in a Tube with a ſpiral Spring: For by the Diviſions of the Index, as it was drawn out more or leſs, I could meaſure the Reſiſtance.

PROB. XIV.

If a Body in a uniform Medium, being uniformly acted on by the Force of Gravity, aſcends or deſcends in a right Line; To find the Times, Velocities, and Spaces deſcribed.

Suppoſe as before $W =$ Weight of the Globe.
$d =$ it's Diameter.
$q =$ it's Denſity.
$p =$ Denſity of the Medium.

$s =$

$s = 16\frac{1}{12}$ Feet the Space through which a heavy Body descends by Gravity in a Second.

$x =$ Space described from the beginning of the Motion.

$t =$ Time of describing x.

$v =$ Velocity at the End of the Time t.

$b =$ Velocity the Body is projected upwards with, if it ascends. Here I measure the Velocity by the Space uniformly described in a Second.

The comparative Weight of the Globe in the Medium will be $\frac{q-p}{q}W$. And we shall find, as in the last Problem $\frac{3pvv}{16dsq}W =$ Resistance of the Globe moving with Velocity v; and consequently $\frac{q-p}{q}W \pm \frac{3pvv}{16dsq}W$ is the Force acting on the Globe according as it ascends or descends; call this Force y. Now $2st' =$ Moment of Velocity generated by Gravity in the Time t': And the Moment of Velocity being as the Force and Moment of Time universally; therefore $2st' : Wt' :: \mp \dot{v} : yt' :: \mp \dot{v} : y\dot{t}$; whence $\dot{t} = \mp \frac{W\dot{v}}{2sy} = \frac{\mp 8dq\dot{v}}{\overline{q-p} \times 16ds \pm 3pvv}$.

Again, let a falling Body describe any small Space $\dot{z}$ at the End of 1 Second; then it will be, $2s$ (Space uniformly described with Velocity $2s$) : 1 Second :: $\dot{z} : \frac{\dot{z}}{2s} =$ Time of describing $\dot{z}$ with Velocity $2s$. And since the Moment of Space $\propto$ Velocity $\times$ Moment of Time universally: therefore $\dot{z} : 2s \times \frac{\dot{z}}{2s} :: \dot{x} : vt'$ $:: \dot{x} : v\dot{t}$. Whence $\dot{x} = v\dot{t} = \frac{\mp 8dqv\dot{v}}{\overline{q-p} \times 16ds \pm 3pvv}$.

Case 1. When the Body ascends, $\dot{t} = \frac{-8dq\dot{v}}{\overline{q-p} \times 16ds + 3pvv}$, and (by Form 5) the Fluent $t = \frac{-8dq \times ,017453}{\sqrt{\overline{q-p} \times 48dsp}}$ $\times$ Degrees in the Arch whose Tangent is $\sqrt{\frac{3pvv}{\overline{q-p} \times 16ds}}$. But when $t = 0$, $v = b$; and the Fluent corrected is $t = \frac{2dq \times ,017453}{\sqrt{\overline{q-p} \times 3dsp}} \times$ Degrees in the Difference of the Arches whose Tangents are $\sqrt{\frac{3pbb}{\overline{q-p} \times 16ds}}$ and $\sqrt{\frac{3pvv}{\overline{q-p} \times 16ds}}$; therefore when $v = 0$, the whole Time of Ascent $t = \frac{2dq \times ,017453}{\sqrt{\overline{q-p} \times 3dsp}} \times$ Degrees in the Arch whose Tangent is $\frac{b}{4}\sqrt{\frac{3p}{\overline{q-p} \times ds}}$.

Likewise $\dot{x} = \frac{-8dqv\dot{v}}{\overline{q-p} \times 16ds + 3pvv}$, and (by Form the 4th) the Fluent $x = \frac{-2.3025 \times 8dq}{6p}$ Log: $\overline{q-p} \times 16ds + 3pvv$: And when duly corrected $x = \frac{4dq \times 2.30258}{3p}$ $\times$ Log: $\frac{\overline{q-p} \times 16ds + 3pbb}{\overline{q-p} \times 16ds + 3pvv}$.

Case 2. When the Globe descends $\dot{t} = \frac{8dq\dot{v}}{\overline{q-p} \times 16ds - 3pvv}$, and (by Form the 6th) $t = \frac{dq \times 2.302585}{\sqrt{\overline{q-p} \times 3dsp}} \times$ Log: $\frac{\sqrt{\overline{q-p} \times 16ds} + \sqrt{3pvv}}{\sqrt{\overline{q-p} \times 16ds} - \sqrt{3pvv}}$.

Also

Also $\dot{x} = \frac{8dqv\dot{v}}{\overline{q-p} \times 16ds - 3pvv}$; whence the Fluent $x = \frac{8dq \times 2.3025}{-6p}$ Log: $\overline{q-p} \times 16ds - 3pvv$: and when corrected $x = \frac{4dq \times 2.302585}{3p} \times$ Log: $\frac{\overline{q-p} \times 16ds}{\overline{q-p} \times 16ds - 3pvv}$.

Cor. 1. The greatest Velocity the Globe can acquire by an infinite Descent is $\sqrt{\frac{q-p}{3p} \times 16ds}$: For when x or t is infinite the Denominator $\overline{q-p} \times 16ds - 3pvv = 0$.

Cor. 2. Let $G = q\sqrt{\frac{d}{\overline{q-p} \times 3ps}}$.

$$H = 4\sqrt{\frac{\overline{q-p} \times ds}{3p}}.$$

$N =$ Number of the Log. $\frac{,4342945}{G}$.

Then $t = 2.30258G \times \text{Log.} \frac{H+v}{H-v}$, or $N = \frac{H+v}{H-v}$, which reduced gives $v = \frac{N-1}{N+1}H$, when the Globe descends.

Cor. 3. $x = 2.302585 \times \frac{4dq}{3p} \times \text{Log:} \frac{HH}{HH-vv}$ $= 2.302585 \times \frac{4dq}{3p} \times \text{Log:} \frac{\overline{N+1}^2}{4N}$, when the Globe descends.

Cor. 4. In like Manner the Velocity and Space may be found from the Time when the Globe ascends.

SCHOLIUM.

The Density q must always exceed p, otherwise the Globe will not gravitate; contrary to the Supposition.

PROB. XV.

To find the Velocity and Resistance of a Globe oscillating in a Cycloid, in a resisting Medium.

FIG. 218. Let Ba be the Arch described in one entire Oscillation, C the lowest Point, and CZ half the whole Cycloidal Arch equal to the Length of the Pendulum: Let the Globe descend from B, and put $CZ = a$, $CB = b$, $BD = x$, the rest as in the last Problem.

Then we shall find $\frac{q-p}{q} W =$ comparative Weight of the Globe in the Medium, and $\frac{3pvv}{16dsq} W$ = Resistance of the Globe moving with the Velocity v, in the Point D, as in the former Problems. Now it is known that CZ is to CD, as the Weight of the Globe $\frac{q-p}{q} W$ is to it's accelerating Gravity at D, which therefore is $\frac{q-p}{q} W \times \frac{b-x}{a}$. Therefore the whole Force by which the Pendulum is urged in D is $\frac{q-p}{aq} bW - \frac{q-p}{aq} Wx - \frac{3pv^2}{16dsq} W = y$. And therefore we shall find (the same as in Prob. XIV.) $\dot{t} = \frac{\dot{v}}{2sy} W$, and $\dot{x} = (v\dot{t} =) \frac{v\dot{v}}{2sy} W$. Put $f = \frac{q-p}{aq} \times 2s$, $g = \frac{3p}{8dq}$, and $h = \frac{q-p}{aq} \times \frac{3p}{4dq} W$: and then $\dot{t} = \frac{\dot{v}}{bf-fx-gvv}$, and $\dot{x} = \frac{v\dot{v}}{bf-fx-gvv}$: And the Fluents will give t and x.

1. Since

1. Since $\dot{x} = \frac{v\dot{v}}{bf - fx - gvv}$, finding the Fluent (by the Help of Form the 4th and Rule 8. Prop. X.) we have $x = \frac{2.302585}{-2g}$ Log: $\overline{bf - fx - gvv + \frac{f}{2g}}$; but when $x = 0$, $v = 0$; therefore the Fluent corrected

is $x = \frac{2.302585}{2g} \times \text{Log:} \frac{bf + \frac{f}{2g}}{bf + \frac{f}{2g} - fx - gvv}$.

Let n=Number belonging to the Logarithm $\frac{2gx}{2.30258}$, and then $n = \frac{2bg + 1}{2bg + 1 - 2gx - \frac{2gg}{f}vv}$, which reduced gives $vv = : \frac{n-1}{n}b + \frac{n-1}{2ng} - x : \times \frac{f}{g}$.

2. Let $z = \frac{3pvv}{16dsq}W$ the Resistance in D, then $vv = \frac{16dsq}{3pW}z$, and $v\dot{v} = \frac{8dsq\dot{z}}{3pW}$; expunge v and $\dot{v}$ out of the Value of $\dot{x}$, and we shall have $\dot{x} = \frac{\dot{z}}{bb - bx - 2gz}$, and the Fluent $x = \frac{2.3025}{-2g}$ Log: $\overline{bb - bx - 2gz + \frac{b}{2g}}$: and when duly corrected $x =$

$\frac{2.302585}{2g} \times \text{Log:} \frac{b + \frac{1}{2g}}{b + \frac{1}{2g} - x - \frac{2gz}{b}}$. Let n = Number of the Logarithm $\frac{2gx}{2.30258}$; then will $n = \frac{b + \frac{1}{2g}}{b + \frac{1}{2g} - x - \frac{2g}{b}z}$, which reduced gives $z = \frac{n-1}{n}b + \frac{n-1}{2gn} - x : \times \text{ into } \frac{b}{2g}$.

COR. 1. In the lowest Point C, $n =$ Number belonging to the Logarithm $\frac{3pb}{2.30258 \times 4dq}$. And there the Velocity $= \sqrt{: \frac{\overline{n-1} \times 4dq}{3p} - b : \times \text{into } \frac{f}{ng}}$, and the Resistance $= : \frac{\overline{n-1} \times 4dq}{3p} - b : \times$ into $\frac{q-p}{naq} W$.

COR. 2. But the Velocity and Resistance are the greatest, when $\dot{z}$ or $\overline{bb - bx - 2gz} \times \dot{x} = 0$, and thence $z = \frac{b-x}{2g} b = \overline{b-x} \times \frac{q-p}{aq} W$.

COR. 3. And therefore the Velocity and Resistance are the greatest when $x = \frac{2.30258 \times 4dq}{3p} \times \text{Log}: \overline{\frac{3pb}{4dq} + 1}:$

For then $z = \overline{b-x} \times \frac{b}{2g} = \frac{n-1}{n} \times \overline{b + \frac{1}{2g} - x} : \times \frac{b}{2g}$; which reduced gives $n = 2bg + 1$, and Log. n or the Number $\frac{2gx}{2.30258} = \text{Log: of } \overline{2bg + 1}$:

SCHOLIUM

If the oscillating Body is not a Globe, then the Proportion of it's Resistance to that of an equal Globe whose Diameter is d, must either be calculated from Prob. XXII. Sect. II. or found by Experiments; let that be as 1 to m: And then we must take $\frac{g}{m}$ instead of g, or $\frac{p}{mdq}$ instead of $\frac{p}{dq}$ in the foregoing Calculations.

PROB.

PROB. XVI.

To find the Density of the Atmosphere at any Height; supposing the Force of Gravity to be as any Power of the Distance from the Earth's Center, and the Density of the Air as the Compression.

Let $r =$ Radius of the Earth.
$x =$ Any Distance from the Center.
$d =$ Density of the Atmosphere at the Earth's Surface.
$z =$ Atmosphere's Density at the Distance x.
$n =$ Exponent of the Law of Gravity.

Since the Density is as the Pressure therefore the Moment of the Density $\propto$ Moment of Pressure, that is $\propto$ Moment of Matter $\times$ Force of Gravity: But Moment of Matter $\propto$ Density $\times$ Moment of Space. Therefore the Moment of Density $\propto$ Density and Moment of Space and Force of Gravity; that is $\dot{z} \propto zx^n\dot{x}$ universally.

Now it is collected from Experiments that the Weight of 1 Foot high of Air at the Earth's Surface is to the Weight or Pressure of the Atmosphere as 1 to $29725 = p$, at a mean Density; therefore, let $\dot{x}$ be any small Height, and it is $p : d :: \dot{x} : \frac{d\dot{x}}{p} =$ Moment of Density at the Earth's Surface. Whence from the foregoing general Proportion, $\frac{d\dot{x}}{p} : dr^n\dot{x} :: -\dot{z} : zx^n\dot{x}$: therefore $\dot{z} = \frac{-x^n z\dot{x}}{pr^n}$, and $\frac{\dot{z}}{z} = \frac{-x^n\dot{x}}{pr^n}$: And (by Form the 1st and 2d) the Fluent

is

is 2.30258 Log: $z := \frac{-x^{n+1}}{\overline{n+1} \times pr^n}$. And duly corrected, $2.302585 \times$ Log: $\frac{z}{d} := \frac{r^{n+1} - x^{n+1}}{\overline{n+1} \times pr^n}$; therefore $z = d \times$ Number of the Log: $\frac{r^{n+1} - x^{n+1}}{2.3025 \times \overline{n+1} \times pr^n}$ $= d \times$ Number belonging to the Log. $\frac{r^{n+1} - x^{n+1}}{68444 \times \overline{n+1} \times r^n}$.

COR. 1. If e be any small Height above the Earth's Surface, then $z = d \times$ Number of the Log. $\frac{-e}{68444}$.

COR. 2. If $n = 1$, then $z = d \times$ Number of the Logarithm $\frac{rr - xx}{68444 \times 2r}$.

COR. 3. If $n = 0$, $z = d \times$ Number of the Logarithm $\frac{r - x}{68444}$.

COR. 4. If $n = -2$, $z = d \times$ Number belonging to the Logarithm $\frac{r \times \overline{r - x}}{68444x}$. In all which e, r and x are supposed to be taken in Feet.

PROB. XVII.

To find the Density of the Atmosphere at any Height; supposing the Force of Gravity to be as any Power of the Distance, and the Compression as any Power of the Density.

Let $r =$ Radius of the Earth.
$d =$ Density at the Earth's Surface.
$x =$ any Distance.

$z =$

$z =$ Density at the Distance x from the Center.
$p =$ a Length of 29725 Feet.
$n =$ Index of the Force.
$m =$ Index of the Density.
$v =$ compressing Force at the Distance x.

Now by the Hypothesis $v \propto z^m$, and Force $\propto x^n$; And $mz^{m-1}\dot{z} \propto$ Moment of Pressure, that is as the Moment of Space and Density and Force of Gravity: that is $z^{m-1}\dot{z} \propto x^n z\dot{x}$, or $z^{m-2}\dot{z} \propto x^n\dot{x}$, universally.

To find the Moment of Density at the Earth's Surface, we have $\dot{v} \propto mz^{m-1}\dot{z}$; therefore (by Prop. II.) $v : \dot{v} :: z^m : mz^{m-1}\dot{z} :: z : m\dot{z}$, therefore $\dot{z} = \frac{z\dot{v}}{mv}$; but at the Earth's Surface $z = d$; and taking any very small Space $\dot{s}$) it will be, $p : v :: \dot{s} : \dot{v} :: \dot{s} : \dot{v}$, and $\frac{\dot{v}}{v} = \frac{\dot{s}}{p}$; whence $\dot{z} = \frac{d\dot{s}}{mp} =$ Fluxion of Density at the Earth's Surface: Therefore from the universal Proportion; $d^{m-2} \times \frac{d\dot{s}}{mp} : r^n\dot{s} :: z^{m-2}\dot{z} : -x^n\dot{x} :: z^{m-2}\dot{z} : -x^n\dot{x}$: Whence $z^{m-2}\dot{z} = \frac{-d^{m-1}x^n\dot{x}}{mpr^n}$; and the Fluent (by Form the 2d) is $\frac{z^{m-1}}{m-1} = \frac{-d^{m-1}x^{n+1}}{\overline{n+1} \times mpr^n}$: And the Fluent corrected is $\frac{z^{m-1} - d^{m-1}}{m-1} = \frac{d^{m-1}}{\overline{n+1} \times mpr^n} \times \overline{r^{n+1} - x^{n+1}}$:

And by Reduction $z^{m-1} = d^{m-1} + \frac{\overline{m-1} \times r}{\overline{n+1} \times mp} d^{m-1} - \frac{\overline{m-1} \times x^{n+1}}{\overline{n+1} \times mpr^n} d^{m-1}$.

PROB. XVIII.

To find the Diameters of the Earth.

FIG. 227. Suppose the Earth to be in the Form of the Spheroid, $APBQ$; AB the Equinoctial, PQ the Axis; and let it's mean Radius $CR=1$, $AC=1+v=a$. And $PC=1-v=e$, let CS be the Conjugate to RC, and RT perpendicular to CS, then by Conics $CS=\sqrt{aa+ee-1}=1+vv=c$, and $RT=\frac{ae}{c}=1-2vv=p$. Here I reject the Powers of v above vv as being very inconsiderable.

In the mean Place R a heavy body falls about 16,0917 Feet in 1 Second; and the versed Sine of the Arch described by R in 1 Second by the Earth's Revolution is ,04, (if $RC=21000000$ Feet); also as $1:\sqrt{\frac{1}{2}}::,04:,0283=$ to the centrifugal Force in R, as 16,0917 represents the Force of Gravity; therefore $16,12=d$, will be the gravitating Force at R if the Earth stands still: And this is nearly the same with that of a Spheroid whose Axis is $2RT$, and Radius of the (Base or) greatest Circle $\sqrt{ac}$; which (by Corol. 2. Prob. XI.) is $2p-\frac{4pp}{3\sqrt{ac}}-\frac{2pp}{5ac\sqrt{ac}}\times\overline{ac-pp}=\frac{2}{3}+\frac{4}{15}v-\frac{187}{30}v^2$, omitting the given Quantities in that Corollary. Also the Force of the Earth at A is nearly the same as a Spheroid whose Axis is AB and Radius of the Base $\sqrt{ae}$, that is (by the same Cor.) $2a-\frac{4aa}{3\sqrt{ae}}+\frac{2aa\times\overline{aa-ae}}{5ae\sqrt{ae}}=\frac{2}{3}+\frac{2}{15}v+\frac{2}{5}vv$. Likewise the Force of the Earth at P is $2e-\frac{4ee}{3a}-\frac{2ee\times\overline{aa-ee}}{5a^3}=\frac{2}{3}+\frac{2}{5}v+4vv$.

For

For the centrifugal Force at the Equinoctial, it is as $\sqrt{\frac{1}{2}} : 1+v :: ,04 : ,05657 \times \overline{1+v} = b + bv$ (by Substitution) $=$ centrifugal Force at A: Also $\frac{2}{3}+\frac{4}{15}v - \frac{187}{30}vv : d :: \frac{2}{3} + \frac{2}{15}v + \frac{2vv}{5} : \overline{1 - \frac{1}{5}v + 10,03vv} \times d$ for the gravitating Force at A, if the Earth stood still: from this take $b+bv$ the centrifugal Force, and we get $\overline{1-\frac{1}{5}v+10,03vv} \times d - b - bv$ for the Force of Gravity at A when the Earth is in Motion.

Let $CD=x$, $CE=y$. Since the Gravity and also the centrifugal Force (which is as the Decrease of Gravity) in A and D, are as a or $1+v$ to x, therefore the gravitating Force of the Earth in D, when the Earth is in Motion will be $\overline{1-\frac{6}{5}v+11.23v^2} \times dx - bx$.

Lastly, $\frac{2}{3}+\frac{4}{15}v - \frac{187}{30}vv : d :: \frac{2}{3}+\frac{2}{5}v + 4vv : \overline{1+\frac{1}{5}v+15.27vv} \times d$ the Force of Gravity at P. And since the Forces in P and E are as e or $1-v$ to y, therefore the Force at E is $\overline{1+\frac{6}{5}v+16,47vv} \times dy$.

Now suppose the Weights of the Columns x and y to be equal; therefore their Moments or Fluxions multiplied into the gravitating Forces at D and E, will be equal; that is $\overline{1-\frac{6}{5}v+11.23vv} \times dx\dot{x} - bx\dot{x} = \overline{1+\frac{6}{5}v+16,47vv} \times dy\dot{y}$, and taking the Fluents, and dividing by $\frac{1}{2}d$, and putting $1+v$ and $\overline{1-v}$ for x and y, there comes out $\overline{1-\frac{6}{5}v+11.23v^2 - \frac{b}{d}} \times \overline{1+v}^2 = \overline{1+\frac{6}{5}v+16.47vv} \times \overline{1-v}^2$; that is $\overline{1-\frac{b}{d}} + \overline{\frac{4}{5}-\frac{2b}{d}} \times v + \overline{9.83-\frac{b}{d}} \times v^2 = 1-\frac{4}{5}v-15.07v^2$, Or $vv + ,0639v = ,000141$, whence $v = ,00213$, and $CA - CP = ,00426$: And therefore if the mean Radius of the Earth be 21000000, then $CA - CP = 89460$ Feet or 17 English Miles nearly: Therefore $AC = 21044730$, and $PC = 20955270$ Feet.

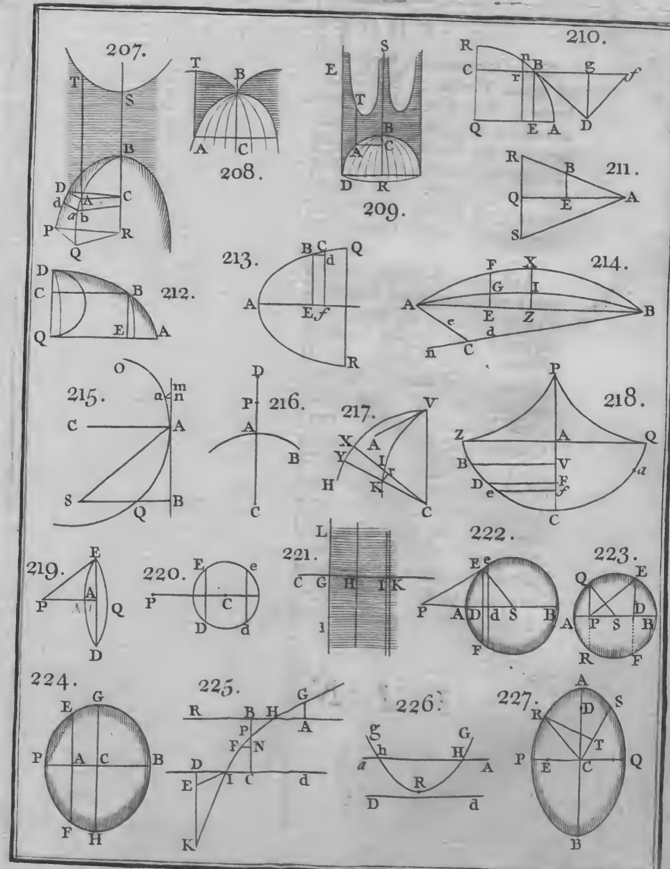
SCHOLIUM.

This Computation ſuppoſes the Earth every where of equal Denſity: But ſince that is not certainly known, nor with what Force a Spheroid accurately attracts a Body when ſituated out of the Axis; nor whether the Earth itſelf is exactly a Spheroid: Theſe Things may render this Solution a little incorrect. If the Earth be more rare towards the Equinoctial than towards the Poles; it's Height at the Equinoctial will be encreaſed in that Proportion.

And now I might proceed to the Calculation of other more difficult Problems, ſuch as *finding the Curves deſcribed by Bodies acted upon by any Laws of Gravity, and moving in Mediums which reſiſt as any Powers of the Velocity; the Motion of the Nodes and Apſides of the Moon; the Preceſſion of the Equinoxes; and ſuch like.* But ſince theſe cannot be diſpatched in a few Words, but often run into long and tedious Calculations, and require a great deal of Room, I ſhall not trouble the Reader with them, eſpecially ſince the Method of purſuing and managing theſe is the very ſame as in thoſe Problems here delivered. And therefore I ſuppoſe, if the Reader underſtands what has been before laid down, he will be able of himſelf to apply *this Doctrine* to the Solution of any other Problem that happens to fall in his way, though more complex, without any further Aſſiſtance.

F I N I S.

The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X. *the end*.

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Note, when *b* is set to the Page, you must reckon from the Bottom.

Pag.	Line	Read
40	16, 17	*for* 4480 *read* 40320.
	1, 2 *b*.	*for* 1116 *read* 1368.
51	1 *b*.	$\overline{x^m \dot{x}}$
104	10	$ax - o = yy - o$.
108	2 *b*.	Ratio of the Space and Time
112	6	break it at
114	15	then $\frac{1}{2}t =$
118	21	Let $AB = r$,
119	2	Quantity b;
170	2 *b*.	*dele* (whence $o = \frac{8a}{27}$);
179	10	and $BD = BD =$
180	2 *b*.	$9au^2 = y^3$,
187	4	$\frac{-c}{2}\sqrt{p - vv}$
195	22	is $NN - N$,
210	6	$\overline{5d - 3v}$, for the
212	7	2.30258ca^3 Log:
213	5	(when $s = o$, $y = a$)
	13	$\sqrt{ay - yy}$
219	10	therefore *on*—
279	7	*ED*, *ed*
284	7 *b*	$yy - bb :: p : q$,

From Page 108 *to Page* 287, *for* α *read* ∝.